SUMMITING HFK(K):

a view from the top and next-to-top gradings

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slides available at:
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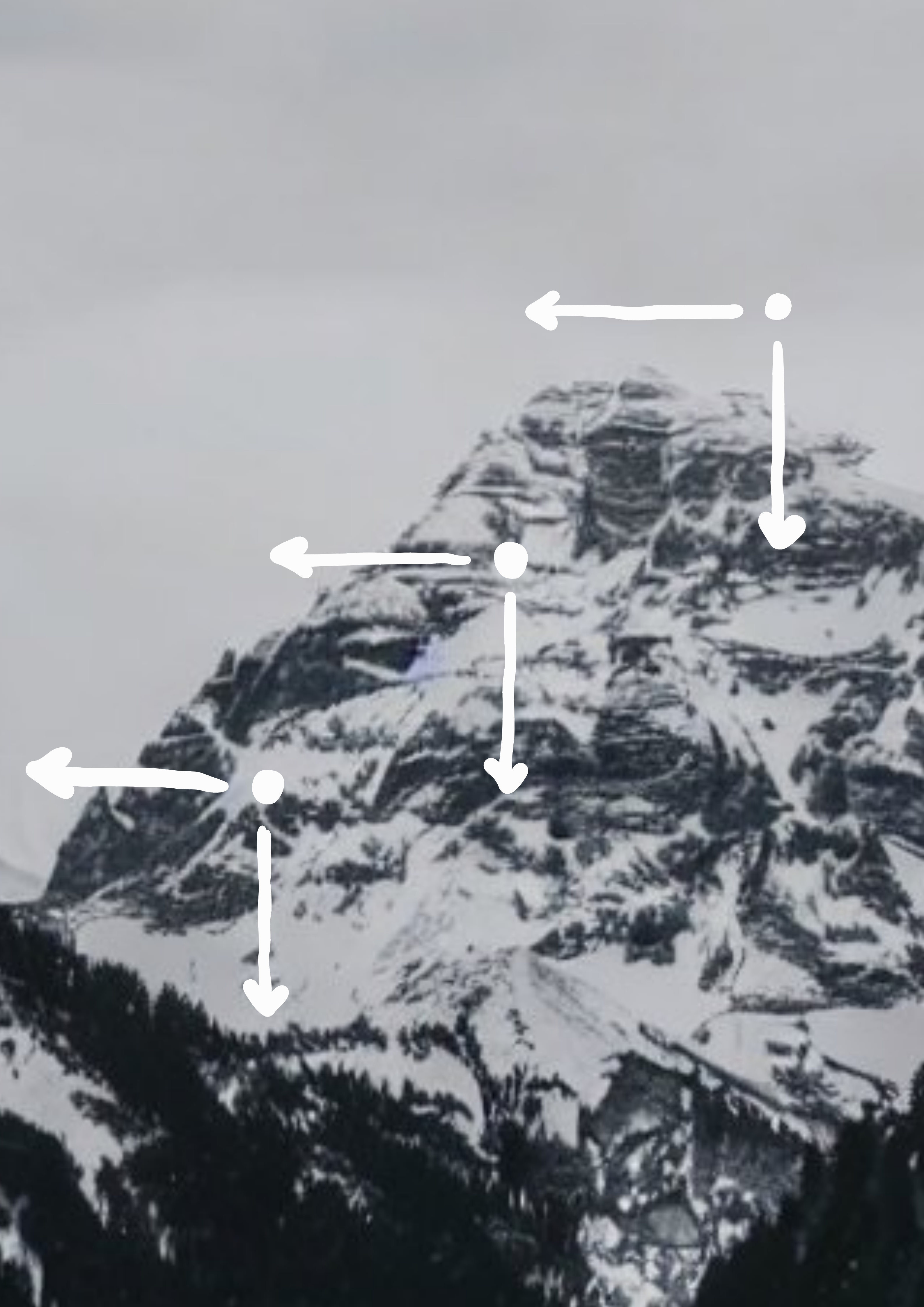


PRELIMINARIES

Throughout, Y is closed, connected, orientable, and of dimension 3.

i.e. $K = \partial\Sigma$ for some orientable surface Σ

Let $K : S^1 \longrightarrow Y$ be a null homologous knot.

KNOT FLOER HOMOLOGY

(independently)
(Rasmussen, Ozsváth & Szabó)

i) associate $K \hookrightarrow Y$ w/ a doubly-pointed Heegaard diagram

$$\mathcal{H}(Y,K) := (\underbrace{\Sigma_g}_{\substack{\text{surface of}\\ \text{genus } g}}, \underbrace{\underbrace{\alpha}, \underbrace{\beta}}_{\substack{\text{systems of } g\\ \text{pairwise disjoint}\\ \text{closed curves}}}, \underbrace{\underbrace{z}, \underbrace{w}}_{\substack{\text{base points}\\ \text{in } \Sigma_g \setminus (\alpha \cup \beta)}})$$

ii) create a chain complex freely generated over the intersection points of Lagrangian tori associated with $\mathcal{H}(K)$.

iii) equip this chain complex with a bigrading and a differential

iv) resulting homology $HFK(K)$ is a bigraded module

HFK° IS INTRINSICALLY INTERESTING...

[Oz-Sz, Ras; '03-'04]

assists with computation of $\mathrm{HF}^\circ \begin{cases} \widehat{\mathrm{HF}} \\ \mathrm{HF}^- \\ \mathrm{HF}^+ \\ \mathrm{HF}^\infty \end{cases}$

[Man-Oz-Sar-Sz; '07 & '09]

can be computed through a combinatorial method (grid homology)

[Man-Sar; '21]

has a Floer stable homotopy type

... & Relates to Other Knot Invariants

[Oz-Sz; '03] "categorifies" the Alexander polynomial

[Oz-Sz; '04] detects the genus of K

[Ni; '07] detects fiberedness of K

& Thurston norm of QHS

rational homology spheres

[Ni, Gh-Sp; '21,'22] & encodes monodromy information

[Al-Ef; '18] yields bounds on the unknotting number

[Oz-Sz; '03] bounds the 4-ball genus of K

[ ] relates to Khovanov homology through a spectral sequence

[Oz-Stip-Sz; '17] [Dai-Hom-Stoffregen-Truong; '21] yields probing homomorphisms into the knot concordance group

etc.

GOALS

I. Sketch definitions of HF° & HFK°

II. Discuss gradings in $\widehat{HFK}(Y,K)$

III. Sketch definition of SFH

IV. Top (Alexander) grading determines knot genus (sketch proof)

V. (Stretch goal) Next-to-top grading encodes monodromy of fibered knots

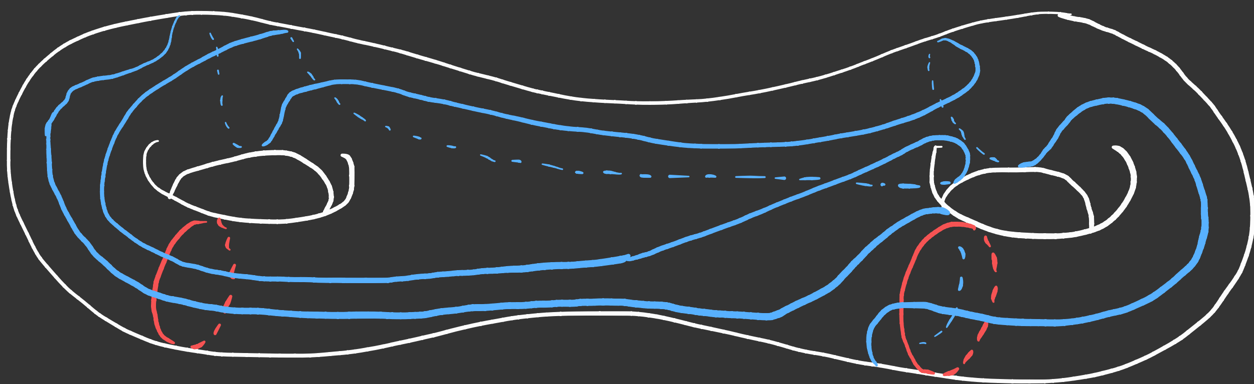
I. 3-Manifold Floer Theories

Let $\mathcal{H} := (\Sigma_g, \alpha, \beta)$ be a

closed surface of genus g

collections of g pairwise disjoint closed curves

Heegaard diagram



Ozsváth & Szabó associate three spaces with $\mathcal{H}$:

- $$\mathrm{Sym}^g(\Sigma_g) := \frac{\prod_g \Sigma_g}{x \sim \sigma(x)} \qquad \sigma \in S_g$$

  (the g-fold symmetric product of Σ_g w/ itself)

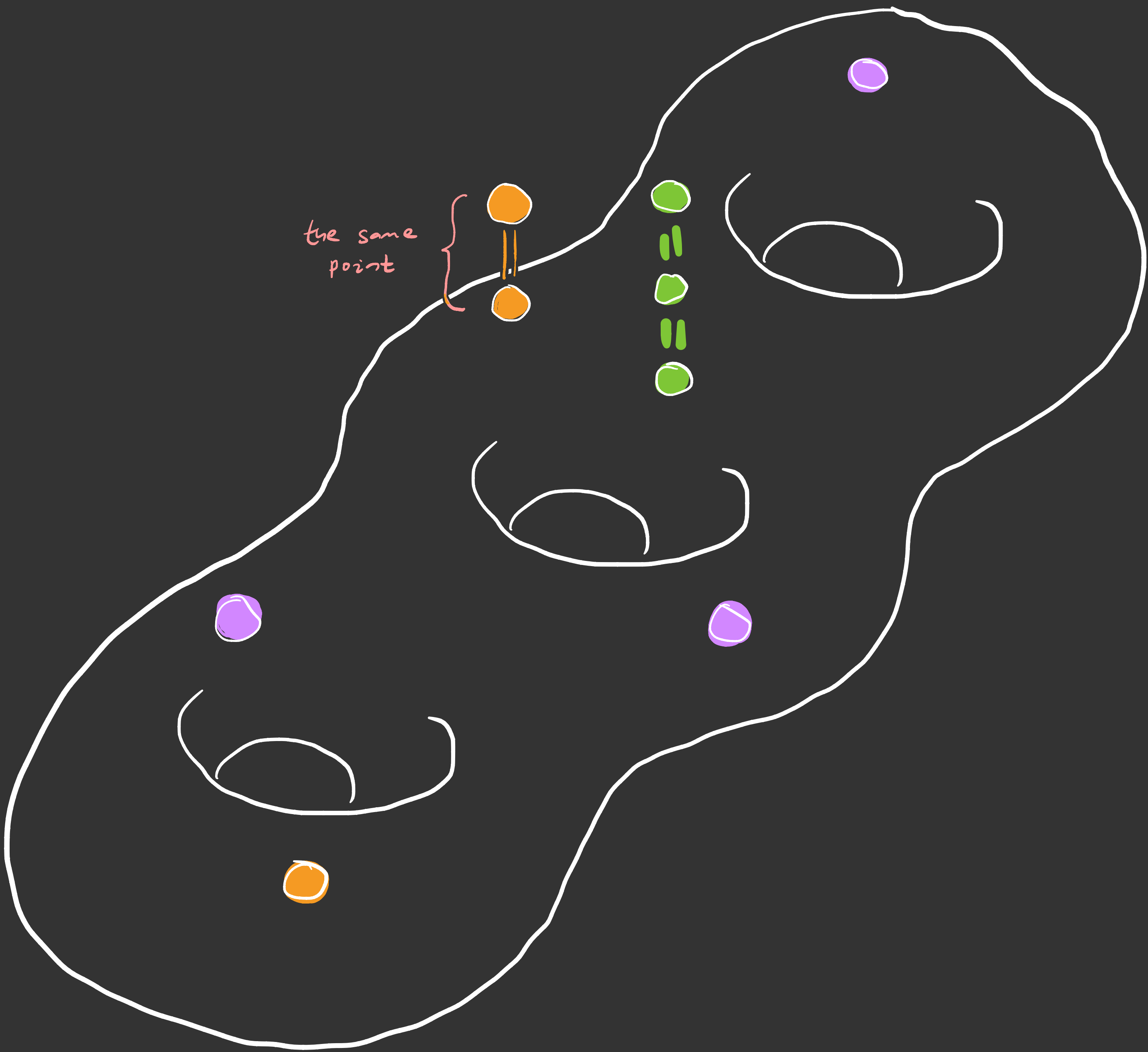
- $\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g$
- $\mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_g$

(Lagrangian tori)

$x \in \mathrm{Sym}^g(\Sigma_g)$ is a finite subset

$$x := \{x_1, \ldots, x_k\} \subset \Sigma_g$$

whose cardinality is at most g.



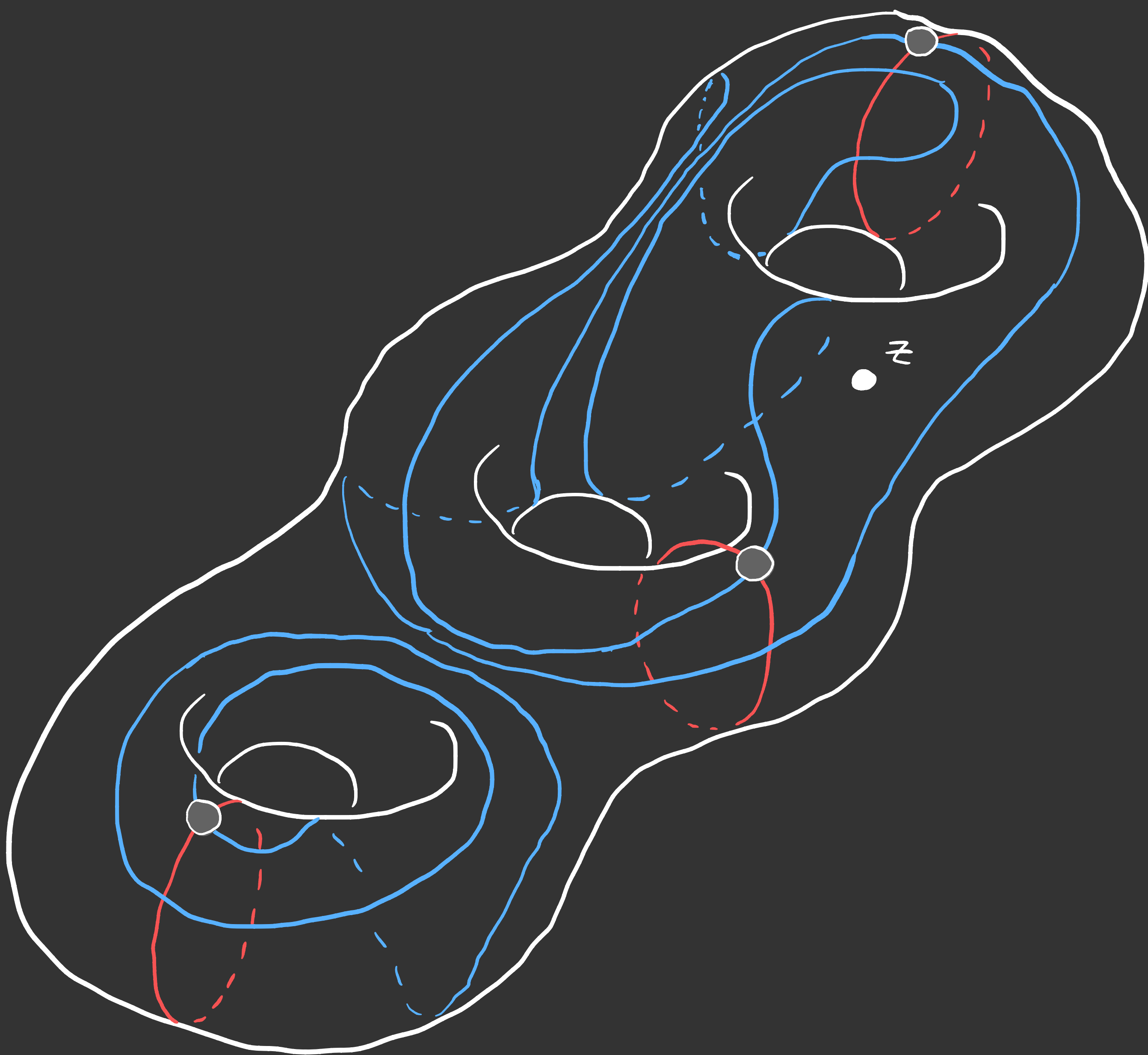
$\in \mathrm{Sym}^3(\Sigma_3)$

$x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a g-element subset

of $\alpha \cap \beta$ such that

$$\forall i \in \{1, \dots, g\}, \; x \cap \alpha_i \cap \beta_j = \{*\}$$

i.e. each α_i and β_j is represented by an intersection point

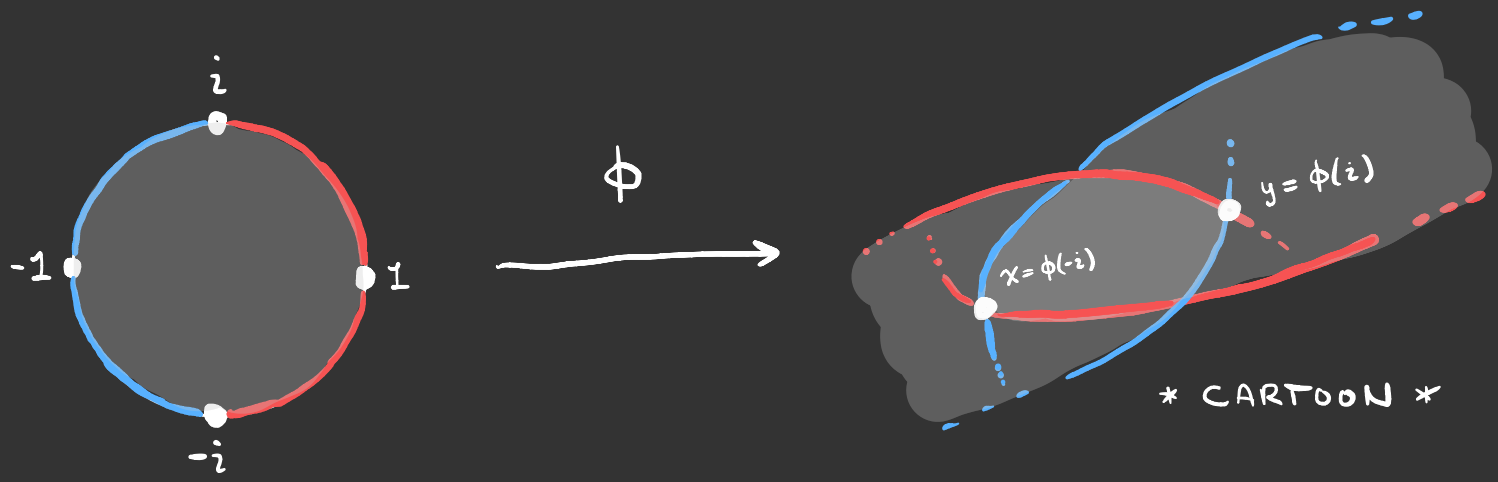


Heegaard Floer (HF°)

Fix a basepoint $z \in \Sigma_g \setminus (\alpha \cup \beta)$,

(this specifies a correspondence between generators of CF & Spin^c structures on M)

build a differential counting holomorphic disks ϕ "connecting" $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$



($\widehat{HF}$) throw out disks having non-zero intersection number with $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$

($HF^\pm_\infty$) record their intersection numbers with powers of a "formal variable" u.

KNOT FLOER ($\widehat{HFK}^\circ$)

- Fix basepoints z & w in $\Sigma_g \setminus (\alpha \cup \beta)$

- Freely generate a $\mathbb{Z}/2\mathbb{Z}[\mathcal{U}, V]$-module

  over $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

- Build a differential which documents the algebraic intersection of holomorphic disks with

$$\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$$

&

$$\{w\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$$

in powers of formal variables $\mathcal{U}$ & V.

Two Basepoints?

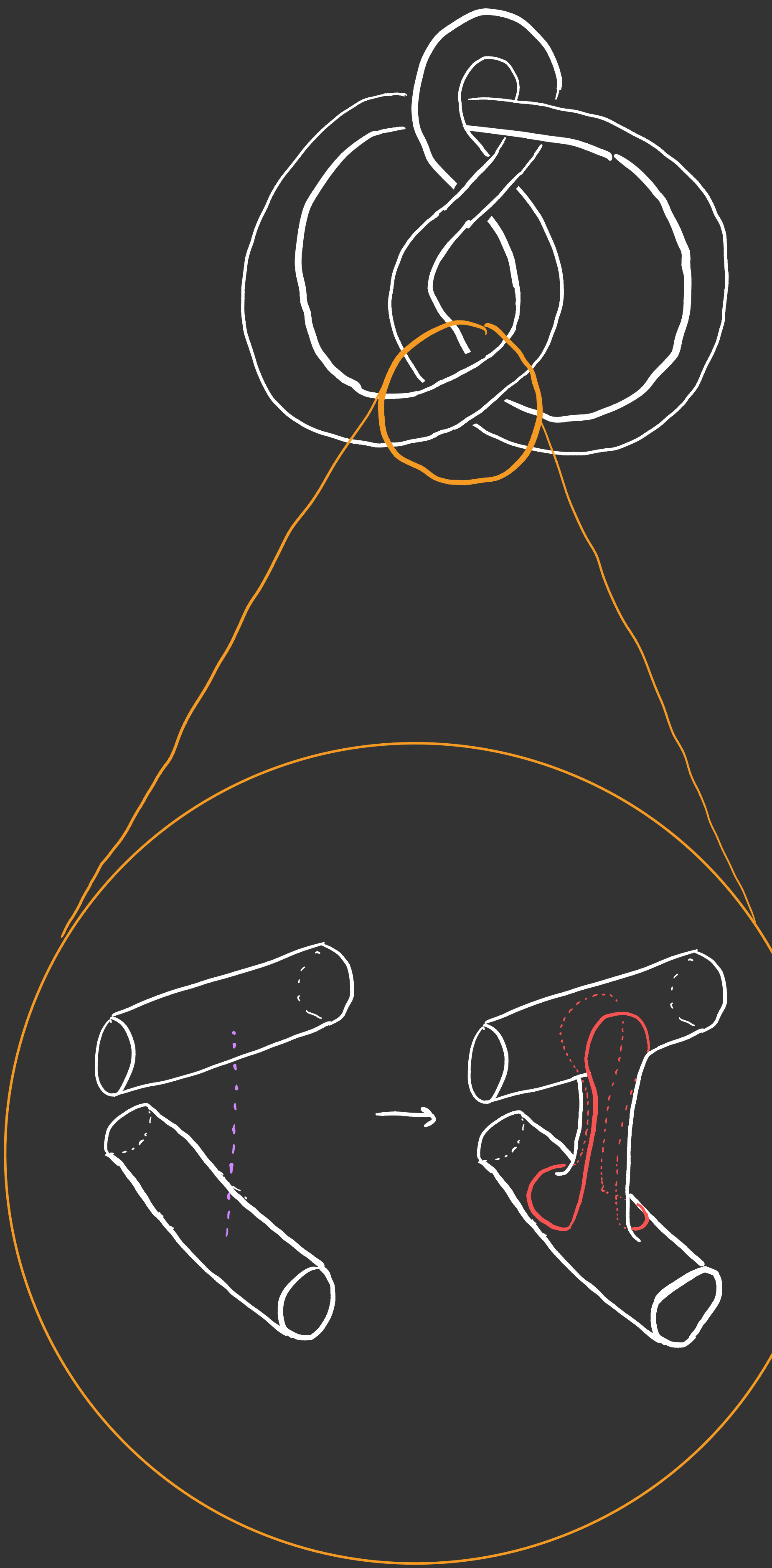
Can realize $K \hookrightarrow Y$ as union of two flowlines connecting the index 0 & index 2 critical points of the Morse function defining a Heegaard splitting for Y^3.

These flow lines will intersect the central surface in two points disjoint from the flow lines connecting other critical points by $\exists!$ - theorem for ODEs.

(Existence & Uniqueness)

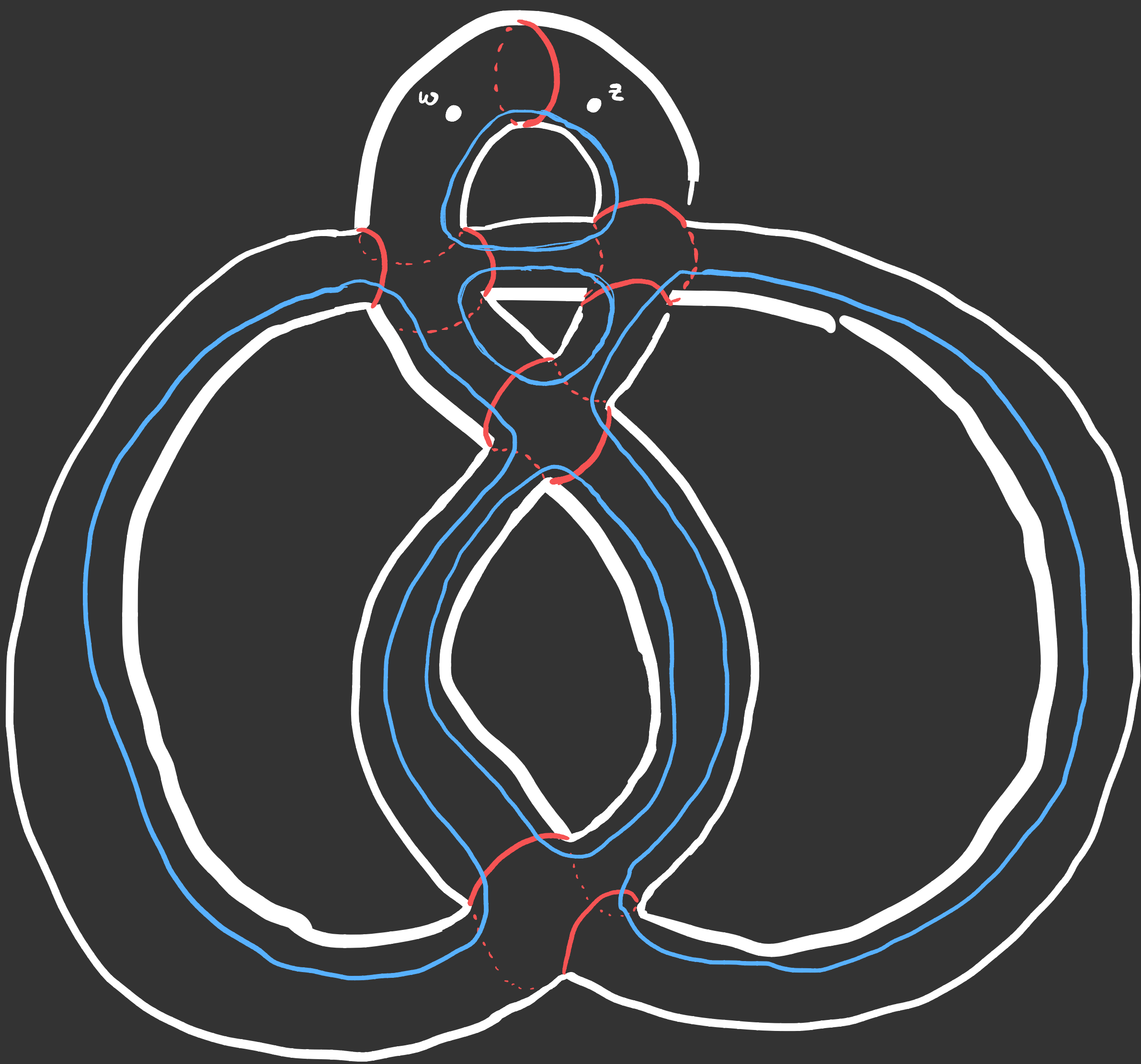
Example $4_1 \subseteq S^3$

1. Take a tubular neighborhood of your knot

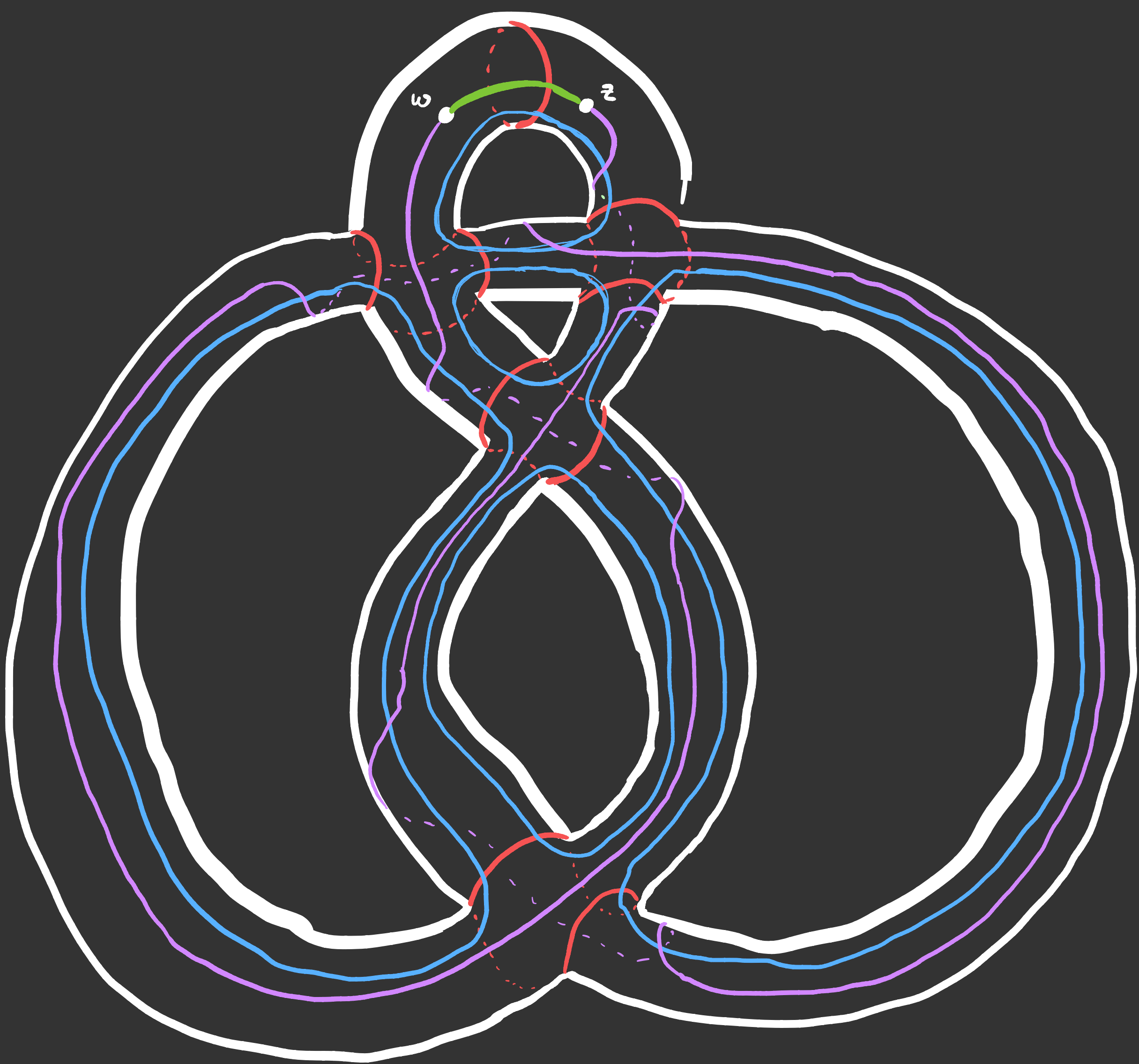


2. Attach tubes & α circles to record crossing information.

3. Separate your base points with another α curve. Lay down β curves.



This faithfully encodes your knot.

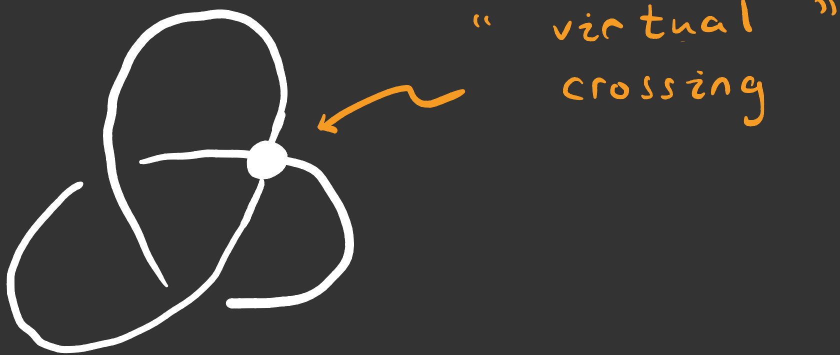


The purple curve lives in the α-handlebody,
so it can intersect the β curves.

The green curve lives in the β-handlebody,
so it can intersect the α curves.

A NAÏVE QUESTION

Kauffman introduced virtual knot diagrams, which encode knots in "thickened surfaces."



quantum topology & "surfaces in 4-manifolds" people seem to be interested in these.

[Is there a similar algorithm for making doubly-pointed Heegaard diagrams for virtual knots?]

II. GRADINGS in HFK

<u>IDEA</u> Partition a ring/module/vector space into levels which are respected by the algebraic structure.

<u>EXAMPLE</u> $H^*(M;R)$ is graded by the cocycle dimension; sums of cocycles in the same graded piece remain in that graded piece, the cup product $\alpha^j \cup \beta^k \in H^{j+k}(M;R)$.

A bigrading on a ring/module/etc. is a pair of gradings.

$\widehat{HFK}^{\circ}$ IS BIGRADED

Endow $\mathbb{Z}/2\mathbb{Z}[\mathcal{U},\mathcal{V}]$ with a bigrading $gr = (gr_{\mathcal{U}}, gr_{\mathcal{V}})$ defined by

$$gr(\mathcal{U}) = (-2,0) \qquad gr(\mathcal{V}) = (0,-2)$$

$CFK(\mathcal{M})$ is a $\mathbb{Z}/2\mathbb{Z}[\mathcal{U},\mathcal{V}]$-module freely generated on points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

This complex is relatively graded
(Prop. 7.5, "Holomorphic disks & 3-manifold Invariants")

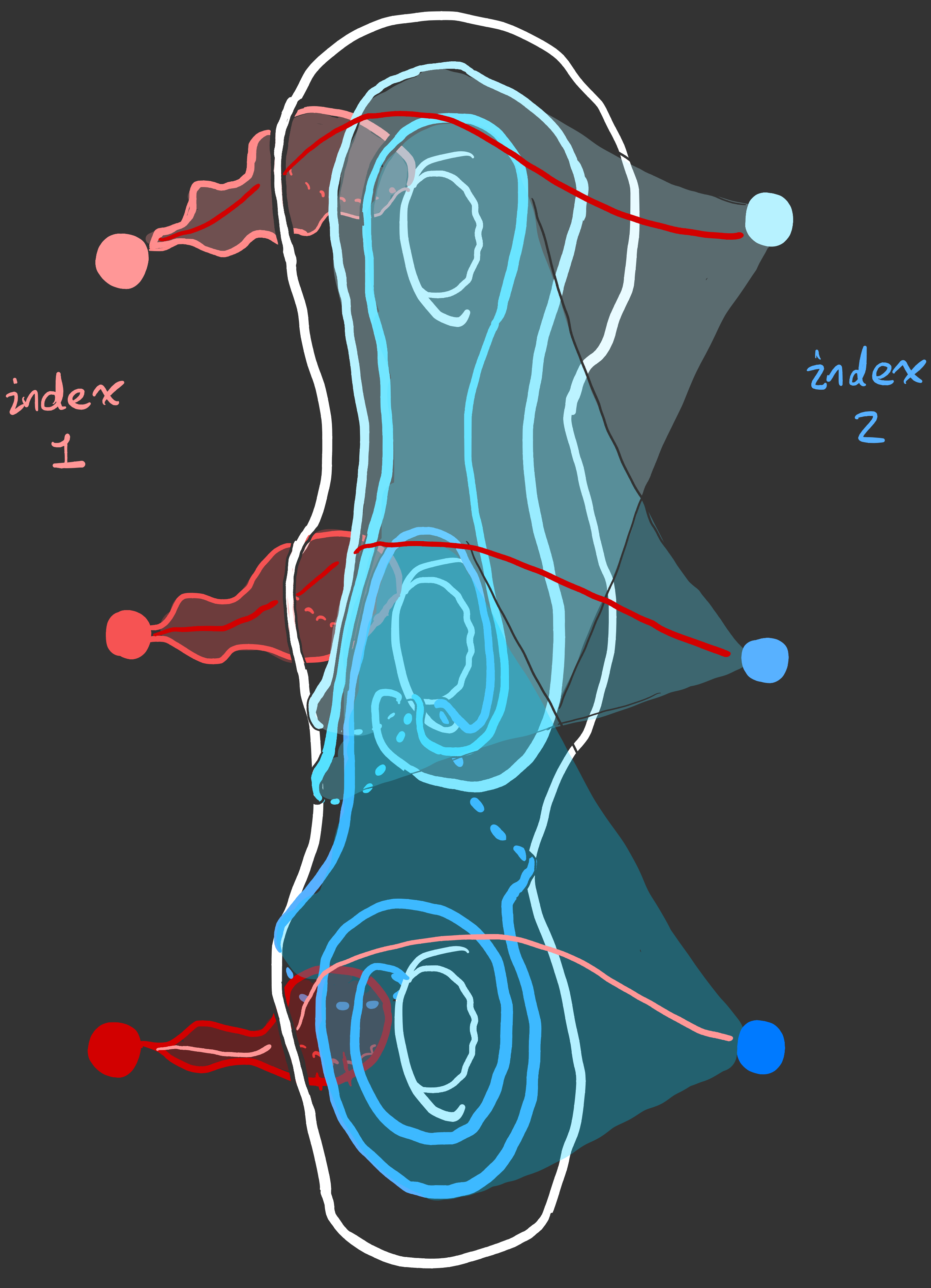
$$\begin{aligned} gr_{\mathcal{U}}(x) - gr_{\mathcal{U}}(y) &= \mu(\phi) - 2n_w(\phi) \\ gr_{\mathcal{V}}(x) - gr_{\mathcal{V}}(y) &= \underbrace{\mu(\phi)}_{\text{the "Maslov index" of } \phi} - 2n_z(\phi) \end{aligned}$$

GRADINGS & Spin^c Structures

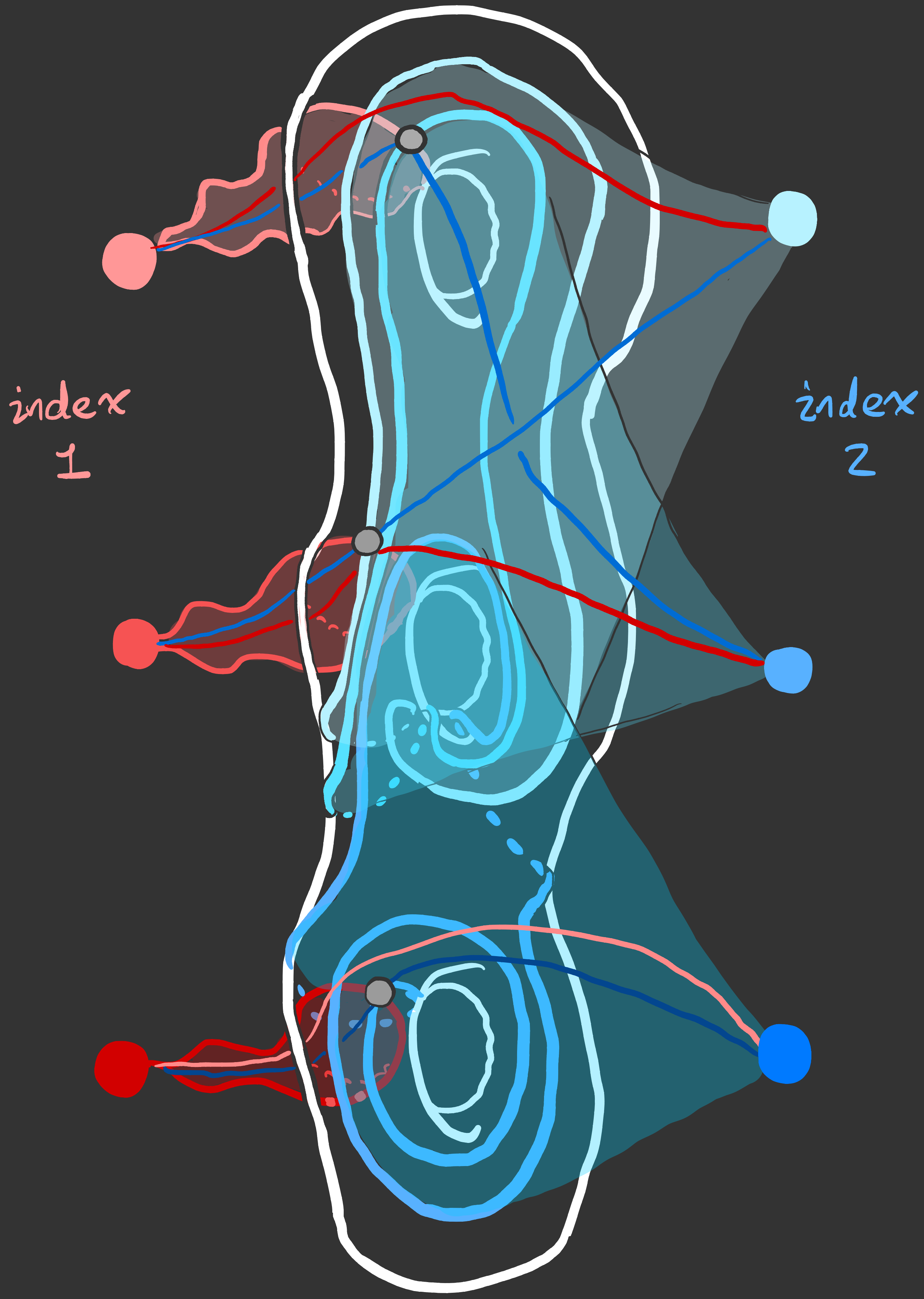
Theorem [Turaev] Let M be a closed, oriented 3-manifold. Then the collection of Spin^c structures on M is in 1-to-1 correspondence with homology classes of non-zero vector fields on M.

Consequence: an association between surfaces, HFK generators, & Spin^c structures

Handle attachments specify arcs connecting index 1 to index 2



$\bullet \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longleftrightarrow$ some closed curve $[\alpha] \in H_1(\mathrm{Sym}^g(\Sigma_g))$



ALEXANDER GRADING

Given $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, its Alexander grading is defined by

$$A(x) := \frac{1}{2}\left(gr_u(x) - gr_v(x)\right)$$

Reminder: gradings behave like logarithms

$$A(u^n v^m x) = \frac{1}{2}\left(gr_u(u^n v^m x) - gr_v(u^n v^m x)\right)$$

$$= \frac{1}{2}\left(\left[gr_u(u^n) + gr_u(v^m) + gr_u(x)\right] - \left[gr_v(u^n) + gr_v(v^m) + gr_v(x)\right]\right)$$

recall: $gr(u) = (gr_u(u), gr_v(u)) = (-2, 0)$

$gr(v) = (gr_v(u), gr_v(v)) = (0, -2)$

$$= \frac{1}{2}\left(\left[-2n + 0 + gr_u(x)\right] - \left[0 - 2m + gr_v(x)\right]\right)$$

$$= m - n + \underbrace{\frac{1}{2}\left(gr_u(x) - gr_v(x)\right)}_{A(x)}$$

We state the differential explicitly:

$$\partial_{\widehat{HFK}} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1}} \underbrace{|\widehat{\mathcal{M}}(\phi)|}_{\in \mathbb{Z}/2\mathbb{Z}} \, U^{n_w(\phi)} V^{n_z(\phi)} y$$

the reduction mod 2 of the cardinality of the compactified moduli space of holomorphic representatives of $\phi \in \pi_2(x,y)$

allows us to define

Exercise The differential sends each object in $\widehat{CFK}$ to another object of the same Alexander grading.

Consequence $\widehat{HFK}$ can be decomposed into a direct sum along levels of the Alexander grading.

$$\widehat{HFK}(M,K) \cong \bigoplus_{s=-\infty}^{\infty} \underbrace{\widehat{HFK}(M,K}_{\substack{\text{portion generated by chains} \\ \text{w/ Alexander grading } i}},s)$$

I'd like to sketch

the proof of ...

<u>Theorem</u> (Oz-Sz, *"Holomorphic Disks & genus bounds"*)

Let $K: S^1 \hookrightarrow S^3$, then the Seifert genus of K, $g(K)$, is the largest $s \in \mathbb{Z}$ such that $\widehat{HFK}(S^3, K, s) \neq 0$.

Ozsváth & Szabó's original proof uses heavy machinery from contact & symplectic topology as well as TQFT properties of HF°

Theorem [Juhász, '07]

We can generalize Os-Sz's theorem to any nullhomologous knot in any rational homology sphere (QHS) using sutured Floer homology.

(By the way, Ni also used these techniques to show HFK detects fiberedness.)

III. Sutured Floer

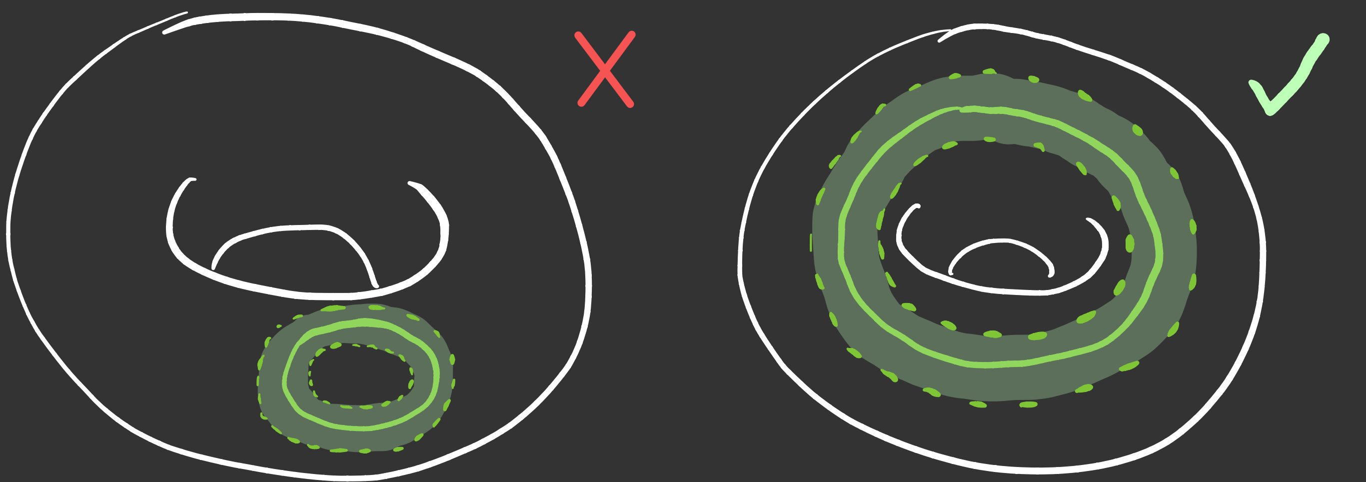
A sutured manifold is a pair

M ~ a compact, oriented 3-manifold w/ boundary

γ ~ a set of pairwise disjoint tori $T(\gamma)$ & annuli $A(\gamma)$ in ∂M.

subject to the conditions

$\boxed{i}$ "$A(\gamma)$'s constituents are algebraically meaningful."



Each annulus in $A(\gamma)$ contains a (homologically) non-trivial closed curve in its interior, called a suture.

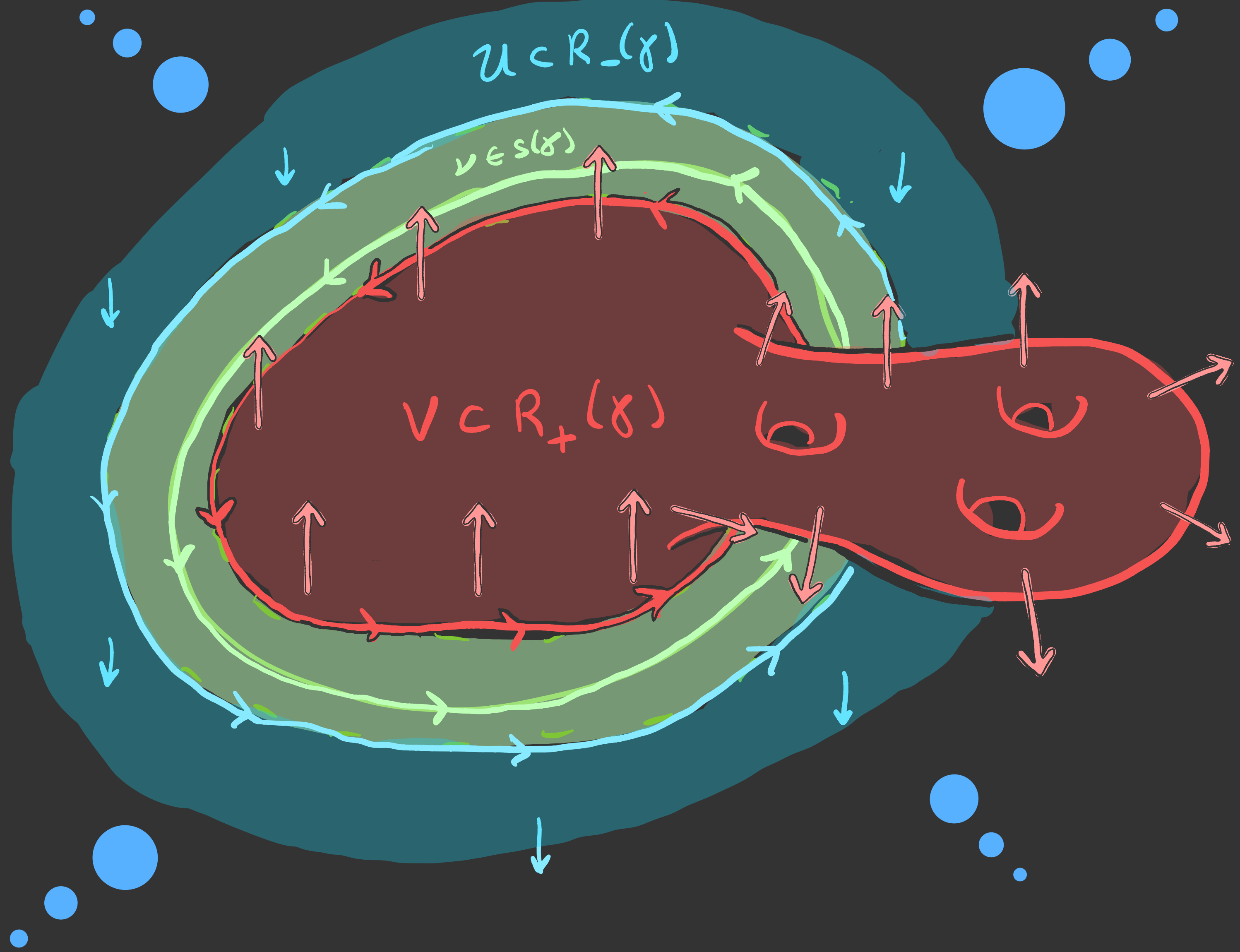
⌜Denote the collection of sutures by $s(\gamma)$.⌟

$\boxed{ii}$ "$s(\gamma)$ orientations determine $\partial M \setminus \operatorname{int}(\gamma)$ orientations"

Define $R(\gamma) := \partial M \setminus \operatorname{int}(\gamma)$

Each component of $R(\gamma)$ is oriented compatibly with the sutures, can decompose $R(\gamma)$ as $R_+(\gamma) \sqcup R_-(\gamma)$

oriented outward   oriented inward



Example

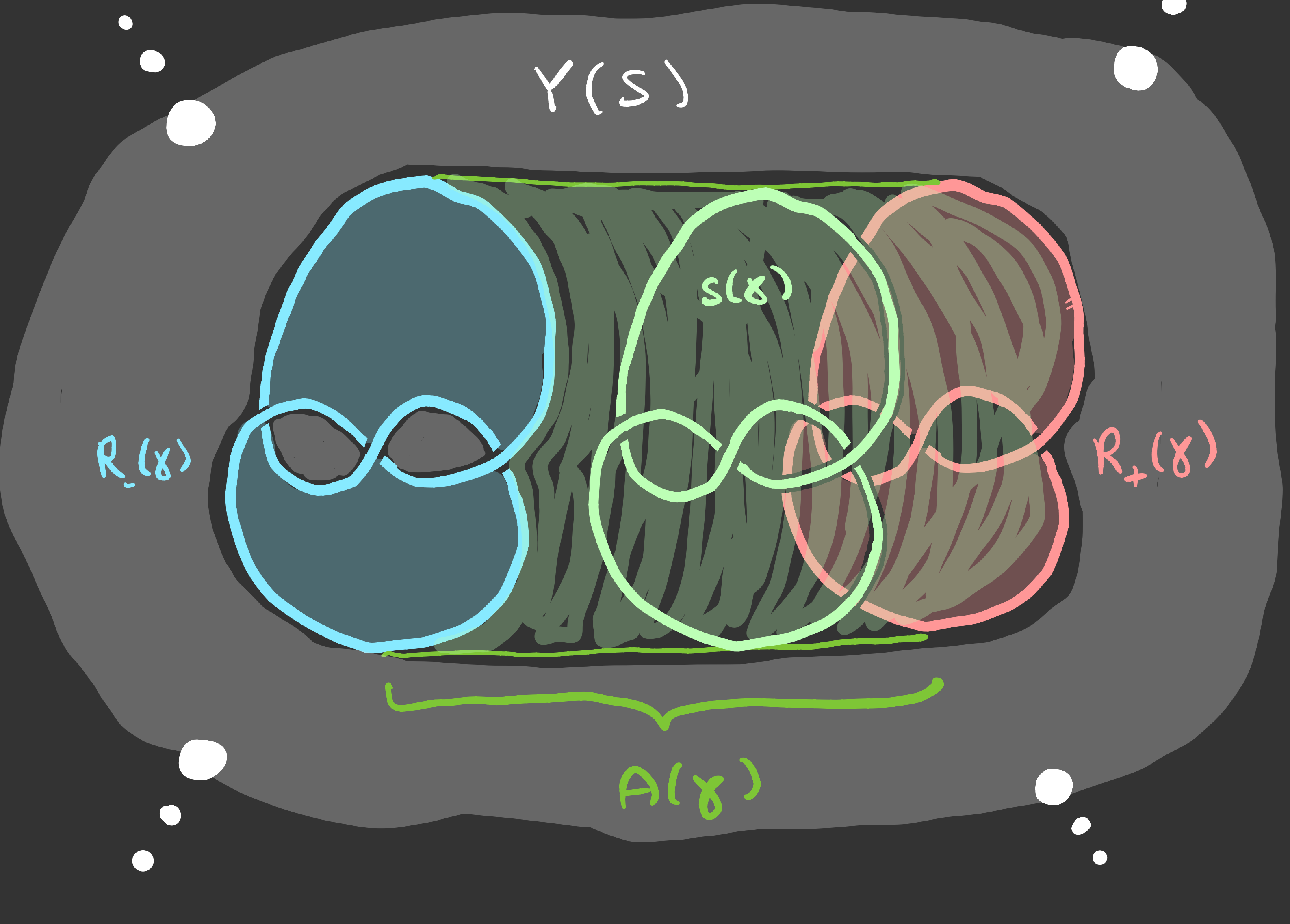
Let Y closed, oriented, connected.
$L \hookrightarrow Y$ a null-homologous link, then if S a Seifert surface for L, define

$$Y(S) := \left(Y \setminus \operatorname{int}(S \times I), \gamma\right)$$

$$A(\gamma) = \partial S \times I \qquad s(\gamma) = \partial S \times \{1/2\}$$

$$T(\gamma) = \emptyset \qquad R(\gamma) = S \times \{0, 1\}$$

* CARTOON *



Remark on $\boxed{ii}$

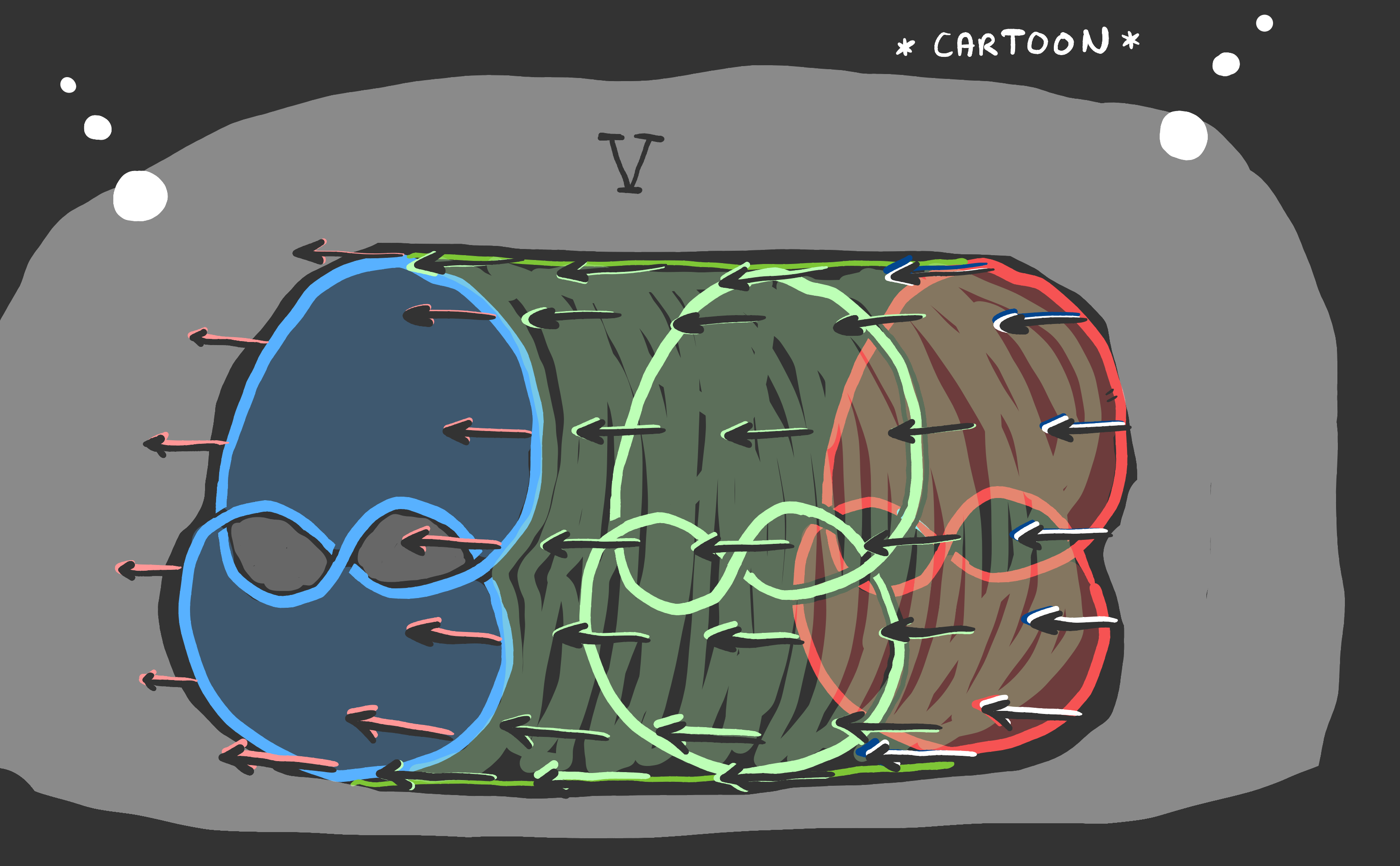
When $\partial M = \emptyset$, we said

Spin^c structures $\overset{1-1}{\longleftrightarrow}$ homology classes of nowhere-vanishing vector fields

For sutured case, fix a nowhere vanishing vector field V on ∂M that agrees w/ orientations on $R_\pm(\gamma)$ & on γ agrees w/ gradient of the height map. $h: s(\gamma)\times I \to I$

Identify Spin^c stuctures w/ homology classes of vector fields (rel ∂) which restrict to V on ∂M.

* CARTOON *



Decomposition along surfaces

The $Y(S)$ example generalizes into a nice "factorization" procedure for sutured manifolds (M, γ)

If $S \subset M$ a properly embedded, oriented surface satisfying certain conditions, then we can decompose Y along S to get (M', γ') w/

$$M' = M \setminus \mathrm{int}(S \times I)$$

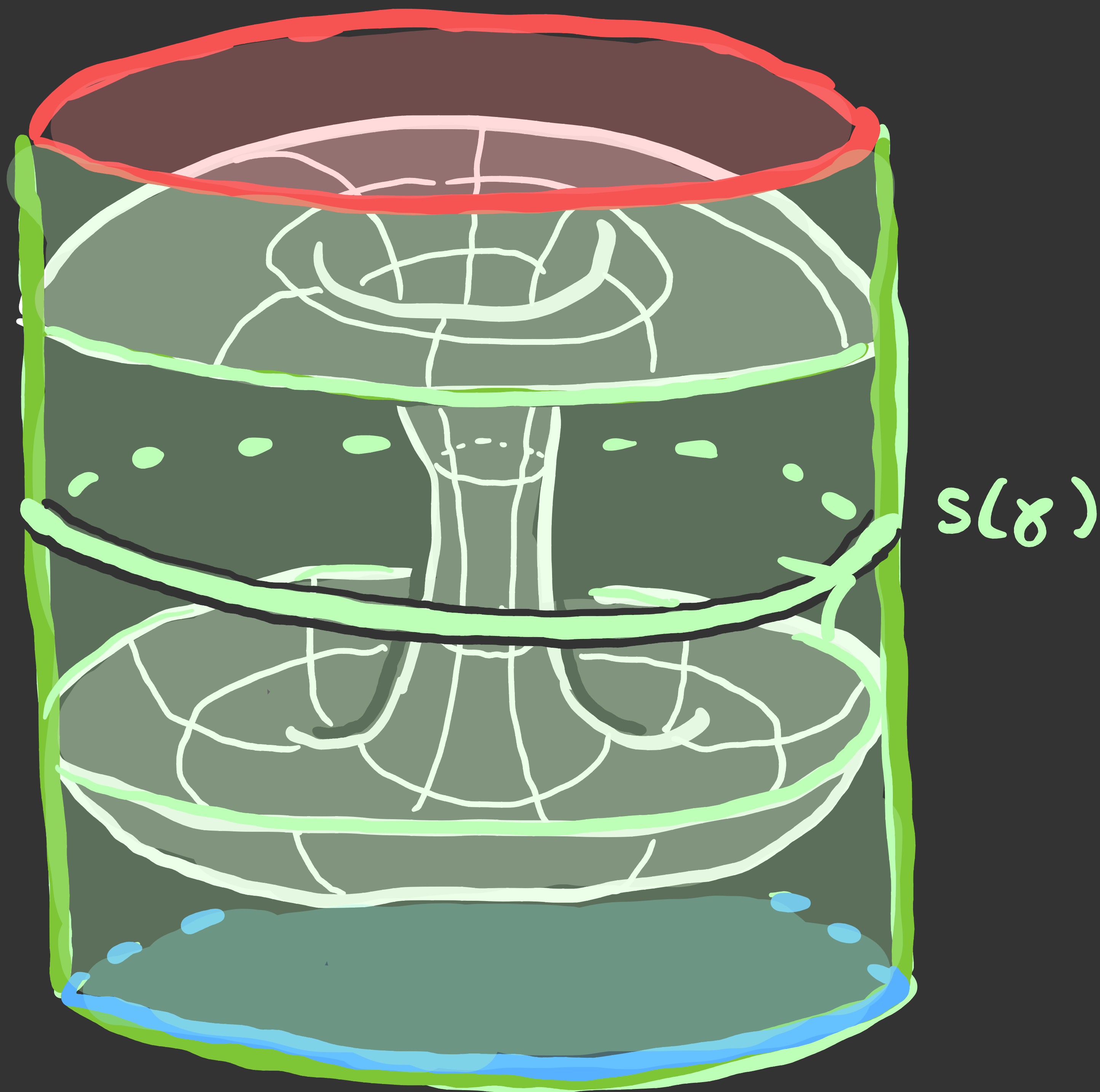
$$\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma))$$

$$R_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+) \setminus \mathrm{Int}(\gamma')$$

$$R_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-) \setminus \mathrm{Int}(\gamma')$$

Where S'_+ / S'_- are components of $\partial N(S) \cap M'$ whose normal vectors point out of / into M'.

Surface admitting a decomposition



(the conditions)

For all components $C \subset S \cap \gamma$ one of the following holds:

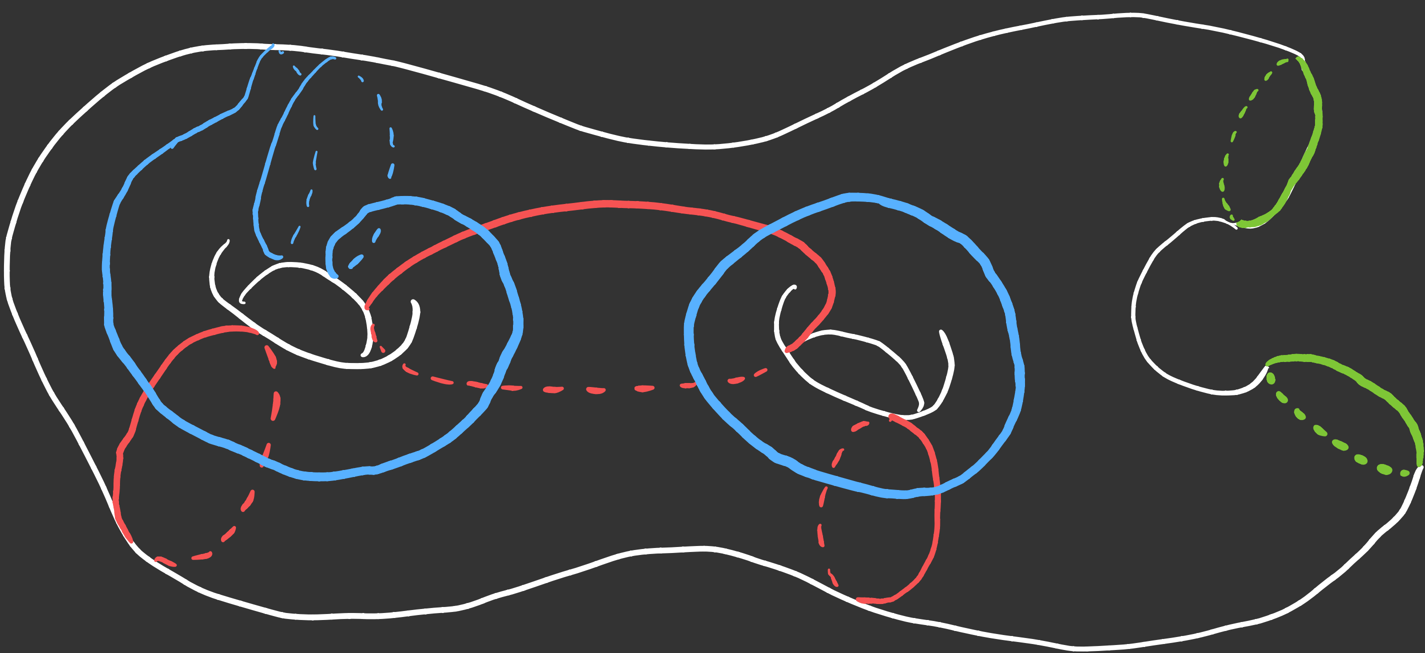
i. C is a properly embedded non-separating arc in γ

ii. C is in a $A(\gamma)$ annulus & represents the same class in $H_1(\gamma)$ as the suture.

iii. All intersections of S w/ a given torus component of γ represent the same class in $H_1(\gamma)$.

A sutured Heegaard diagram is a surface Σ with boundary & collections of pairwise disjoint closed curves

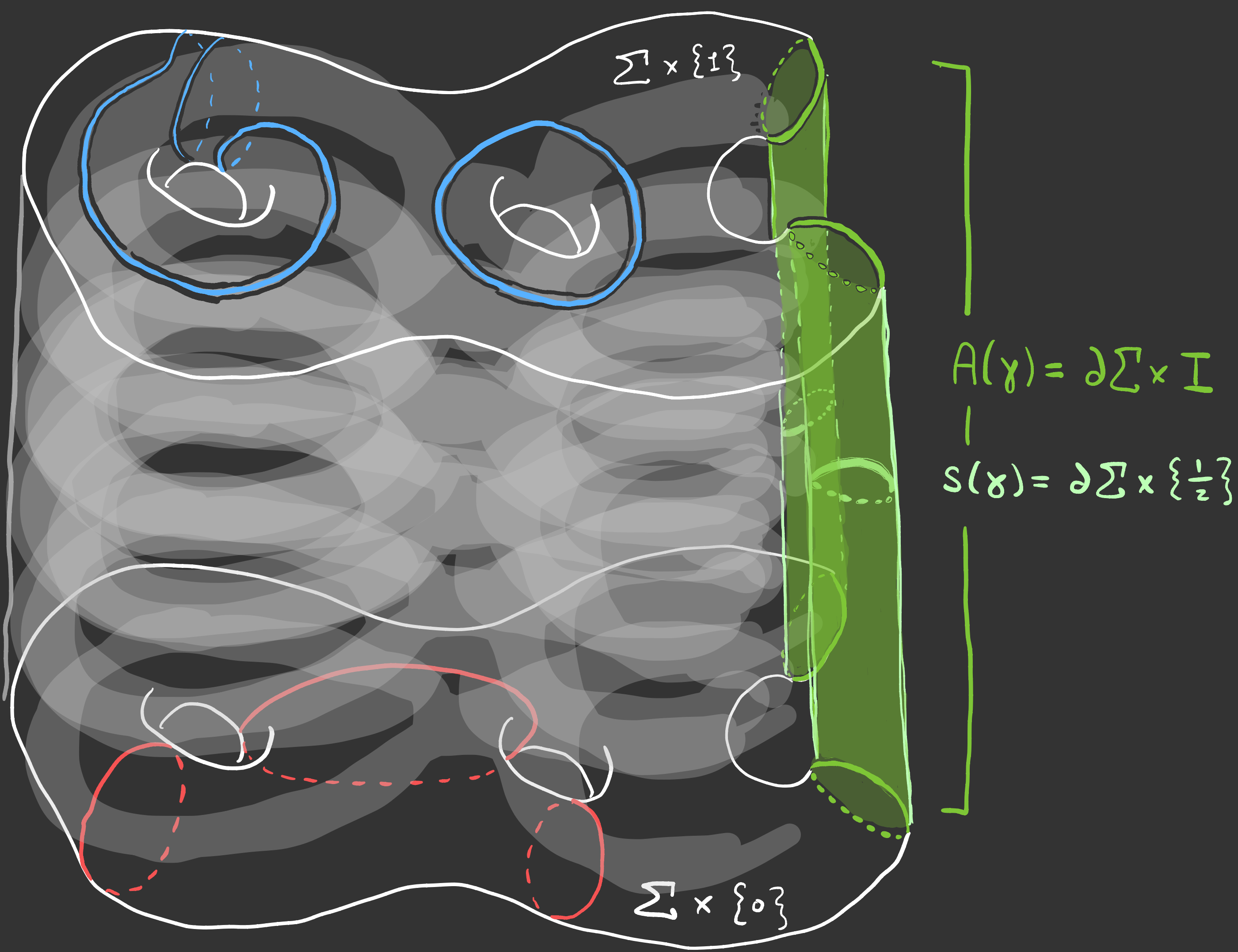
$\alpha = \{\alpha_1, \ldots, \alpha_n\}$ & $\beta = \{\beta_1, \ldots, \beta_m\}$



Note we do not require $n = m$

Sutured Heegaard diagrams encode sutured 3-manifolds

attach 3-dimensional 2-handles along components of $\beta \times \{1\}$



attach 3-dimensional 2-handles along components of $\alpha \times \{0\}$

* CARTOON *

All balanced sutured manifolds have sutured diagrams & diagrams for the same manifold are related by generalized Heegaard moves.

BALANCED MANIFOLDS / DIAGRAMS

A sutured manifold is balanced if:

i. all components of M have boundary

ii. $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$

iii. each component of ∂M has a component of $A(\gamma)$ ($\Rightarrow T(\gamma) = \emptyset$)

A sutured Heegaard diagram is balanced if:

i. $|\pi_0(\alpha)| = |\pi_0(\beta)|$

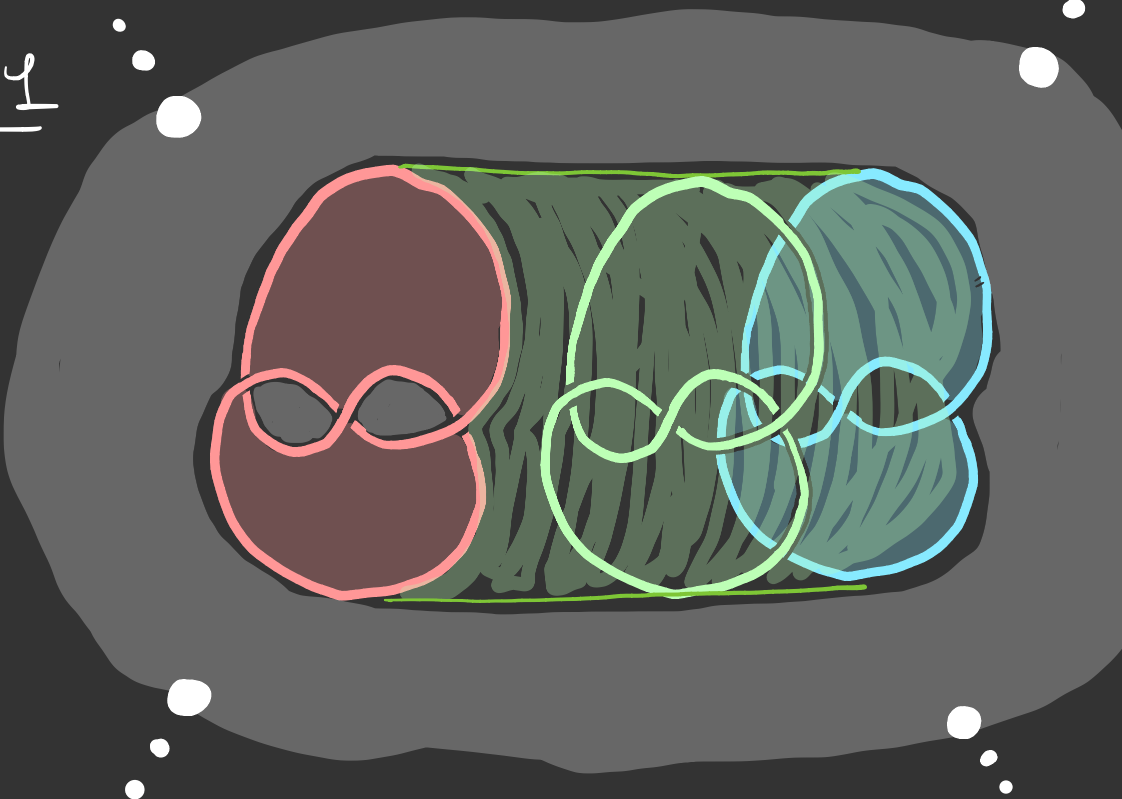
ii. The maps induced by inclusion

$$\pi_0(\partial\Sigma) \longrightarrow \pi_0(\Sigma \setminus \alpha)$$
$$\pi_0(\partial\Sigma) \longrightarrow \pi_0(\Sigma \setminus \beta)$$

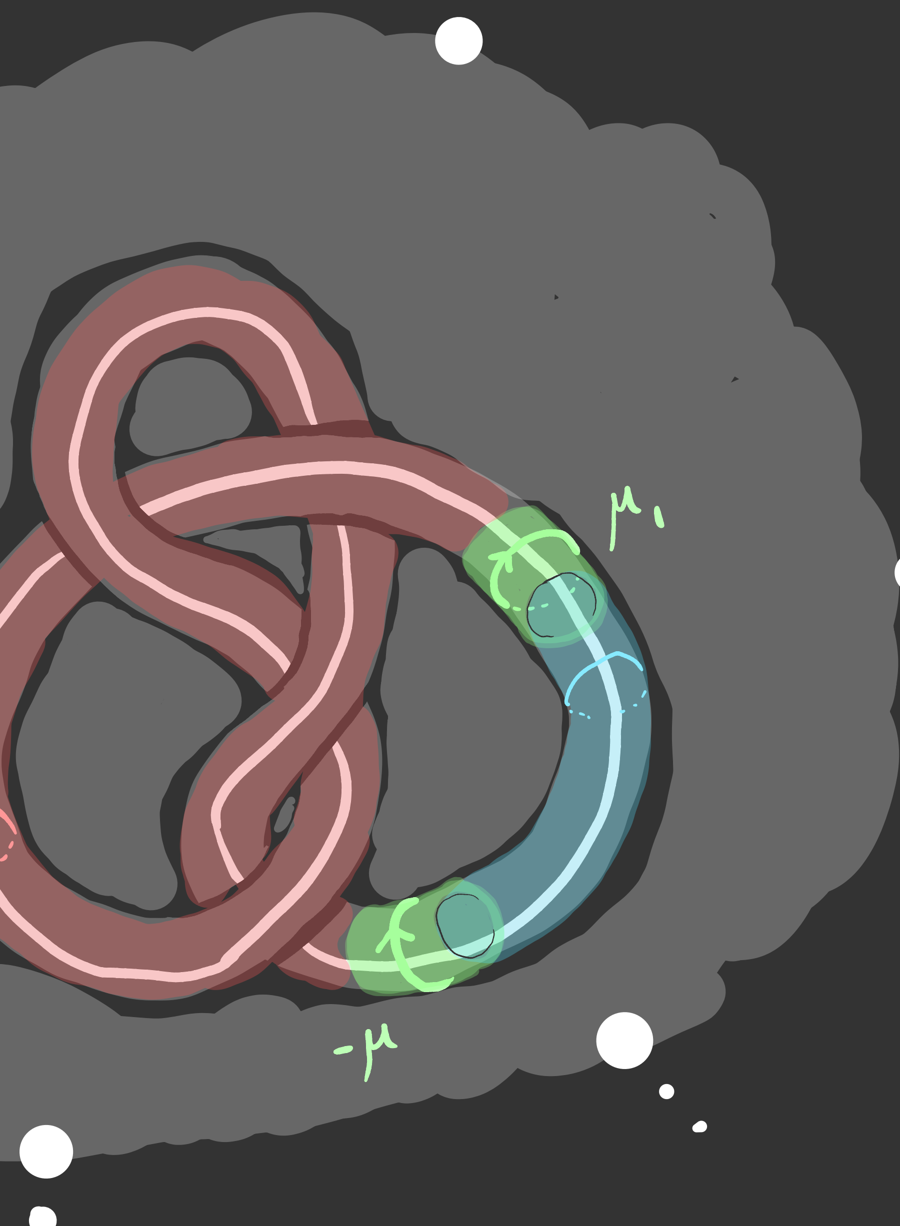
are surjective.

Example 1

$Y(S)$ is (strongly) balanced



Example 2



$Y \setminus N(K)$

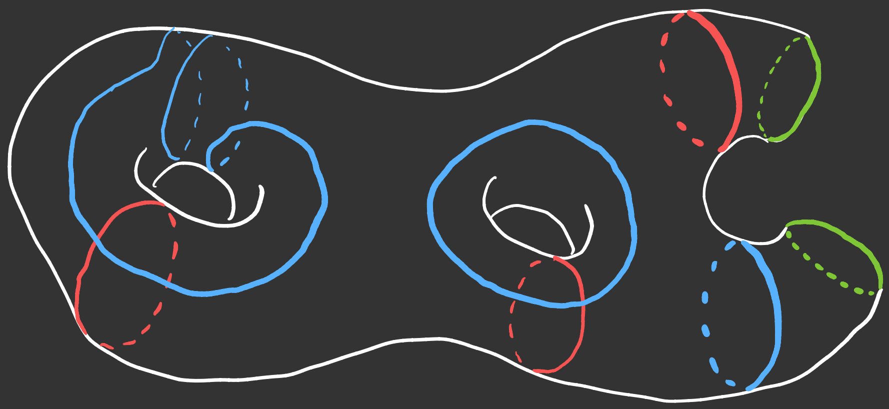
$A(\gamma) := (\mu_1 \sqcup -\mu_2) \times I$

$T(\gamma) = \emptyset$

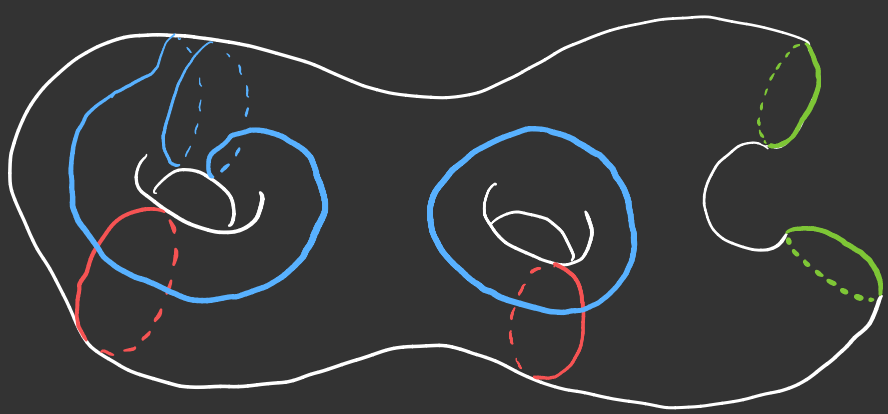
$s(\gamma) := \mu_1 \sqcup -\mu_2$

BALANCED DIAGRAMS

Condition ii. tells us our components of ∂M shouldn't be ghettoized by α & β.



not balanced



balanced

Taut Sutured Manifolds

We say (M, γ) is taut if

- M is irreducible

  "every embedded 2-sphere bounds an embedded 3-ball"

- $R(\gamma)$ is minimal with respect to the Thurston (pseudo-) norm

$$\left[\begin{array}{l} \|\cdot\|_{Th}: H_2(M; \mathbb{R}) \longrightarrow \mathbb{R} \\ \|\alpha\|_{Th} := \min\limits_{\substack{\Sigma \hookrightarrow M \\ [\Sigma] = \alpha}} (-\chi(\Sigma)) \end{array}\right]$$

Such manifolds & their relationships w/ taut foliations were studied by Thurston, Gabai, etc.

SUTURED FLOER (SFH)
(Juhász, '07)

Given a balanced (M, γ), there is always a balanced diagram encoding it.

For such diagrams, we can...

(i) construct Lagrangian tori & generate a chain group from their intersection points

(ii) create a differential counting the ways intersection points can be connected by holomorphic disks

IV. $\widehat{HFK}$ detects knot genus

<u>Sketch</u> We saw we can associate to a Seifert surface $S \subset Y$ a balanced sutured manifold

$$Y(S) := (Y \setminus \mathrm{int}(S \times I), \gamma)$$

We will use this to prove

(a) $\widehat{HFK}(Y,K)$ is trivial in Alexander gradings exceeding $g(K)$.

(b) $\widehat{HFK}(Y,K)$ is non-zero at Alexander grading $g(K)$

(a)

this is sufficient due to some nice results concerning $SFH(Y_1 \# Y_2)$.

Suppose $Y \setminus K$ is irreducible & S a Seifert surface for K with $g(S) > g(K)$.

Note $Y(S)$ will be balanced but not taut.

Argument proceeds directly from two lemmas:

Lemma 1 [Ni] IF (M, γ) is balanced, irreducible, & not taut, then $SFH(M,\gamma) \cong 0$

$$\underbrace{0 \cong SFH(Y(S))}_{\text{Lemma 1 [Ni]}} \cong \widehat{HFK}(Y, K, [S], g(S))$$

Lemma 2

closed connected oriented 3-manifold

For S a Seifert surface in Y we have

$$SFH(Y(S)) \cong \widehat{HFK}(Y, K, [S], g(S))$$

Sketch of Lemma 1 [Ni]

M irreducible & (M,γ) not taut means one of $R_\pm(\gamma)$ is either

- compressible
- does not realize the Thurston norm of its homology class.

In either case, we can decompose (M,γ) along a surface S such that

$$\chi(S) \geq \chi(R_{+\text{ or }-}(\gamma))$$

whichever is giving you problems

The result will be two connected sutured manifolds

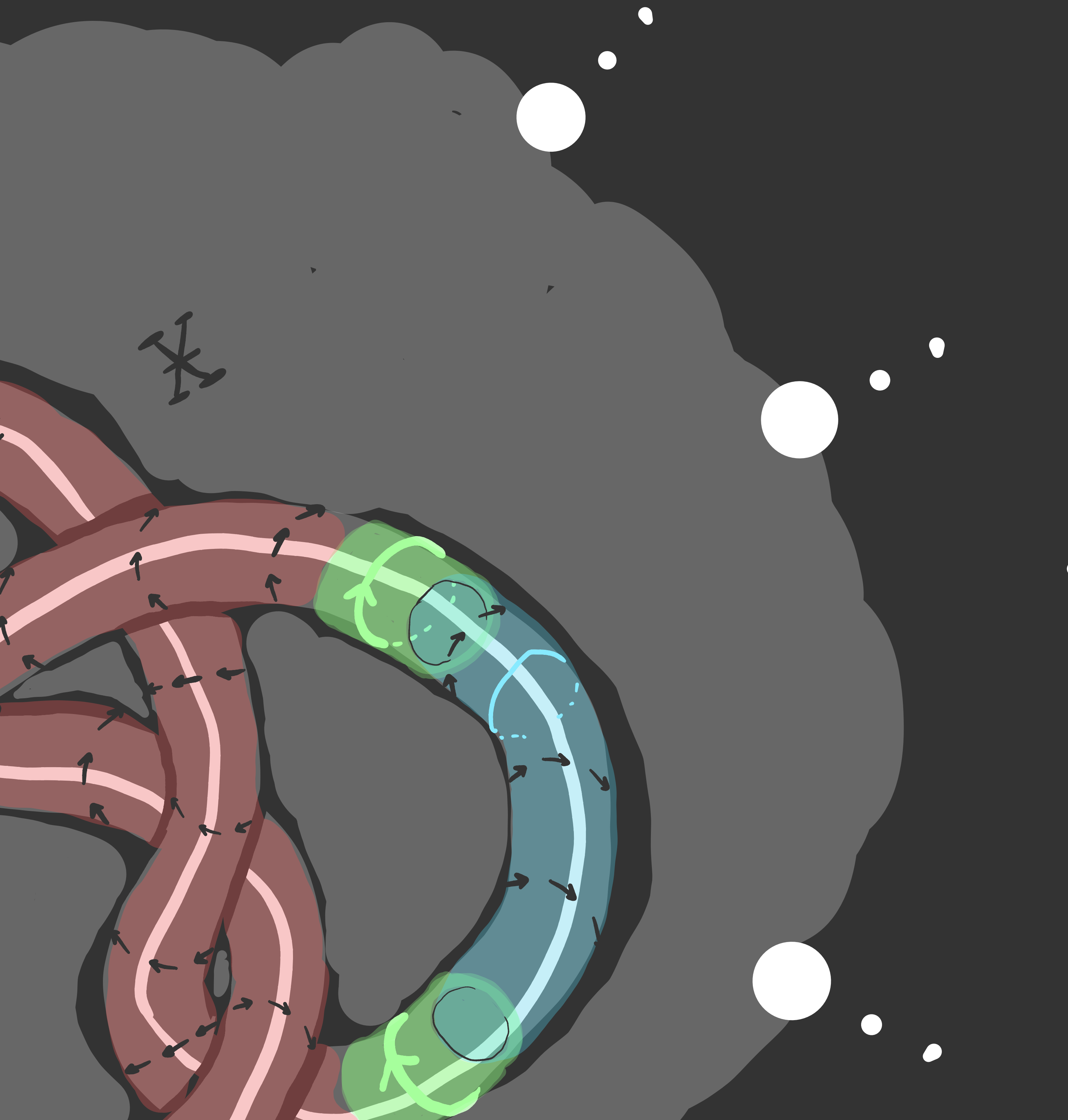
$$(M_+, \gamma_+) \ \& \ (M_-, \gamma_-)$$

There's an explicit Morse function $f_+ \cup f_-$ which can be modified into a self-indexing one for which the corresponding (balanced) diagram has $\alpha \cap \beta = \emptyset$, hence there are no SFH generators!

Lemma 2

Recall $Y \setminus N(K)$ is balanced.

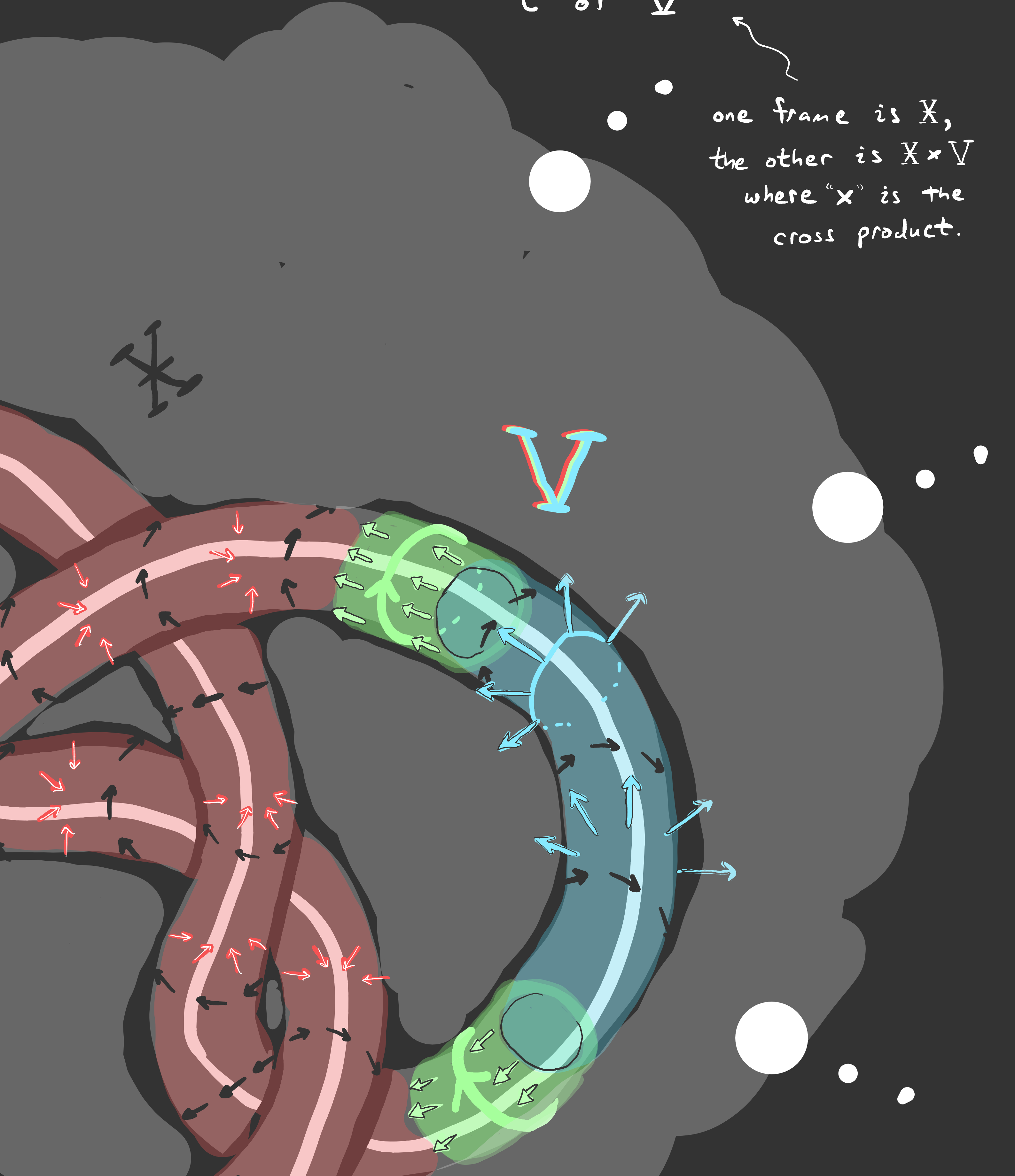
Let X a unit vector field on $\partial N(K)$ pointing in the μ_1 direction.



Recall we fixed a V on ∂M.

Note $X \subset V^{\perp}$, giving a trivialization t of $V^{\perp}$

one frame is X, the other is $X \times V$ where "$\times$" is the cross product.



There's a surface decomposition

$$Y \setminus N(K) \overset{S}{\rightsquigarrow} Y(S)$$

strongly balanced

Juhász showed that when we have such a decomposition,* then

$$SFH(Y(S)) = \bigoplus_{\substack{\mathfrak{s} \in Spin^c(Y \setminus N(K)) \\ c_1(\mathfrak{s}, t) \frown [S] = c(S,t)}} SFH(Y \setminus N(K), \mathfrak{s})$$

$c_1(\mathfrak{s}, t) \in H^2(Y(S), \partial Y(S); \mathbb{Z})$ is the relative Euler class of $\mathfrak{s}$ & t

a term aggregating

- the Euler characteristic of S
- the rotation of t about the positive unit vector field on ∂S.
- projecting the positive unit normal vector field to ∂S to $V^\perp$, its rotation about t.

PUNCHLINE

$$c(S,t) = -2g(S)$$

$$\Downarrow$$

$$\bigoplus_{\substack{\mathfrak{s} \in \mathrm{Spin}^c(Y \setminus N(K)) \\ c_1(\mathfrak{s},t) \frown [S] = -2g(S)}} SFH(Y \setminus N(K), \mathfrak{s}) \cong \widehat{HFK}(Y, K, [S], g(S))$$

Not quite sure how this part works just yet.

Juhász cites

Oz-Sz "Holomorphic disks, link invariats, & multivariable Alexander Polynomial"

A sutured manifold hierarchy is a "tower" of decompositions

$$(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \gamma_n)$$

where $(M_n, \gamma_n) = (R \times I, \partial R \times I)$

i.e. (M_n, γ_n) is a "product sutured manifold"

LEMMA 1

Every taut, balanced sutured manifold (M, γ) admits a hierarchy

LEMMA 2

When S an open surface in (M, γ) admitting a decomposition $(M, \gamma) \xrightarrow{S} (M', \gamma')$, then

$$SFH(M', \gamma') \cong \bigoplus_{\substack{\mathfrak{s} \in Spin^c(M) \\ \mathfrak{s} \text{ is "outer" to } S}} SFH(M, \gamma, \mathfrak{s}) \leq SFH(M, \gamma)$$

LEMMA 3

If $(M, \gamma) = (R \times I, \partial R \times I)$ for some surface R, then $SFH(M, \gamma) \cong \mathbb{Z}$

Let Σ be genus minimizing for $K \hookrightarrow Y$, then $Y(\Sigma)$ is taut & Lemma 1 tells us $Y(\Sigma)$ has a hierarchy

$$Y(\Sigma) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \gamma_n)$$

Lemma 2 & Lemma 3 tell us

$$SFH(Y(\Sigma)) \geq SFH(M_n, \gamma_n) \cong \mathbb{Z}$$

so using the conclusion of (a)

$$SFH(Y(\Sigma)) \cong \widehat{HFK}(Y, K, [\Sigma], g(K))$$

and we're done!

Second-from-top Grading

Theorem (Ni, 2021)

Let Y a closed, oriented 3-manifold & $K \subset Y$ is a hyperbolic fibered knot with fiber F & monodromy φ.

If $\operatorname{rank}(\widehat{HFK}(Y, K, [F], g(F)-1)) = 1$ then φ is freely isotopic to a pseudo-Anosov map w/out fixed points.

We say a map is pseudo-Anosov if there is a pair of measured foliations $\mathcal{F}_1$ & $\mathcal{F}_2$ such that

i. leaves in $\mathcal{F}_1$ are transverse to leaves in $\mathcal{F}_2$
ii. $\mathcal{F}_1$ & $\mathcal{F}_2$ are stable under φ
iii. With respect to the length measure, φ "stretches" $\mathcal{F}_1$ & "squishes" $\mathcal{F}_2$.

What else could it tell us?

Thank You!

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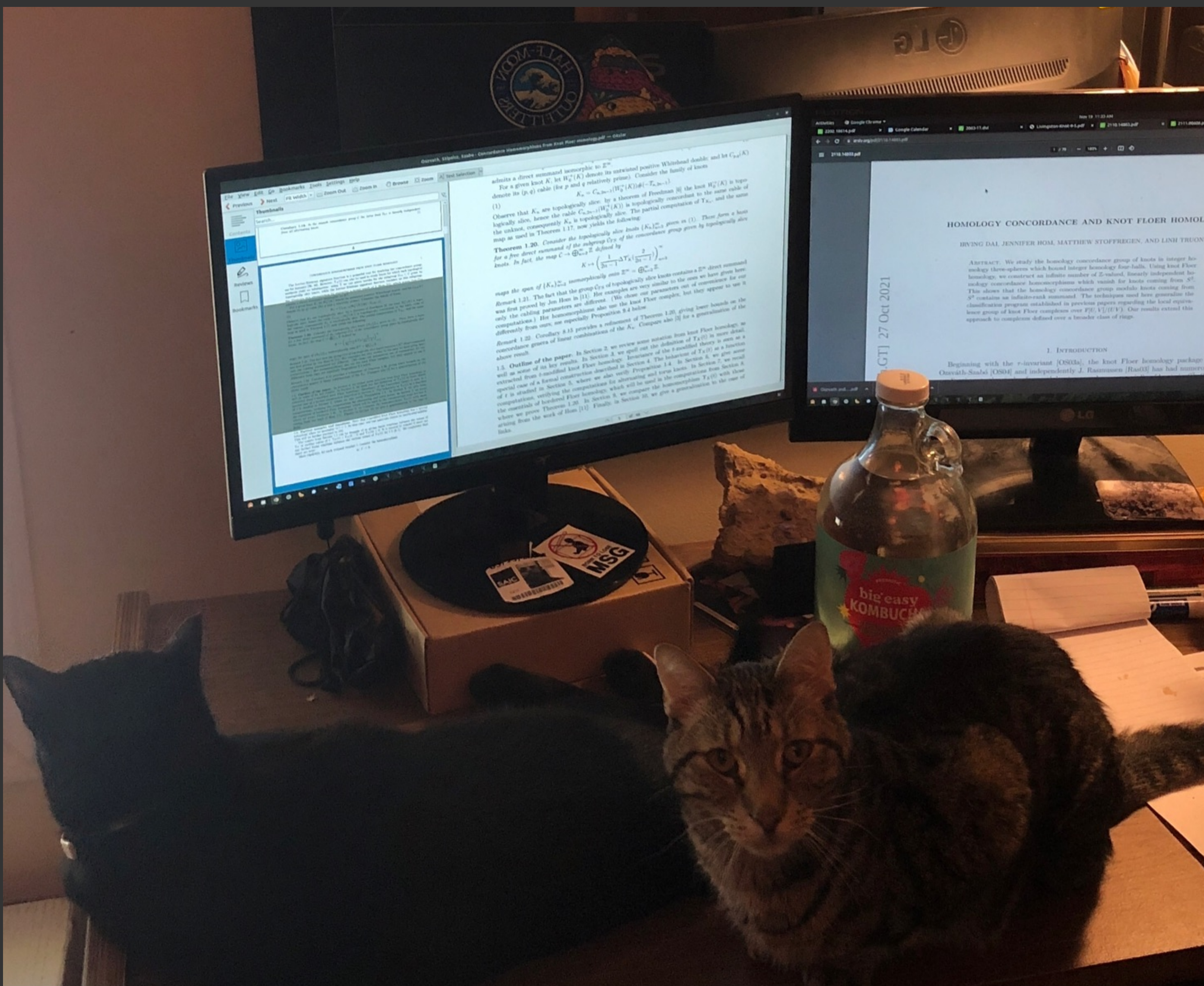

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