An expository talk on:

Slice torus invariants & genus minimizing cobordisms

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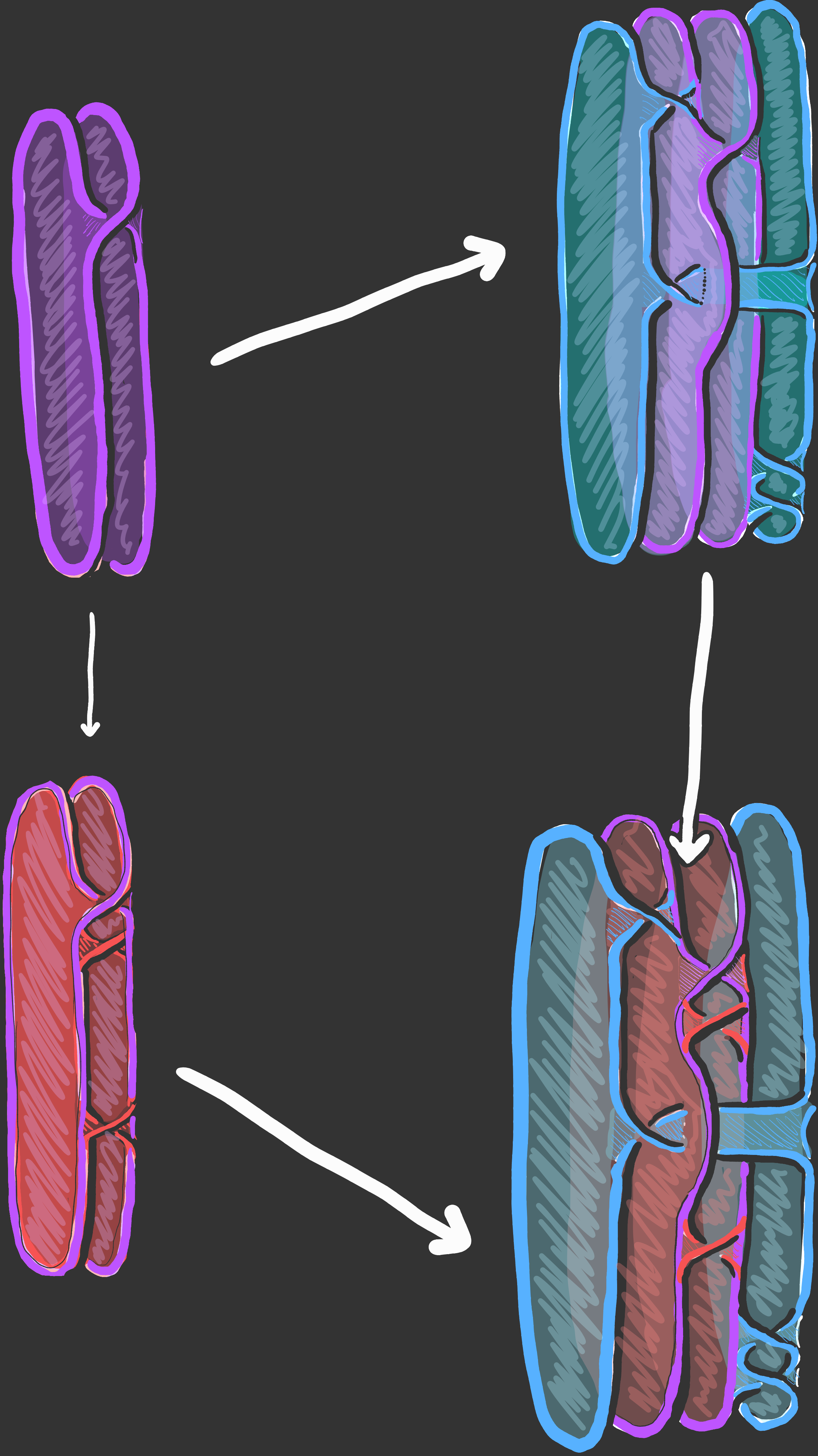
slides @ garydunkerley.github.io



outline

§1. Review & motivation

- smooth 4-genus
- smooth concordance

§2. Rudolph surfaces & quasipositivity

§3. Squeezed knots

[Feller, Lewark, Lobb '21, '22]

§4. Slice torus & "stable" knot invariants

[Jung, Kang, Kim '21]

§5. Turaev genus

§1. review & motivation

Everything smooth, locally flat, embedded in S^3

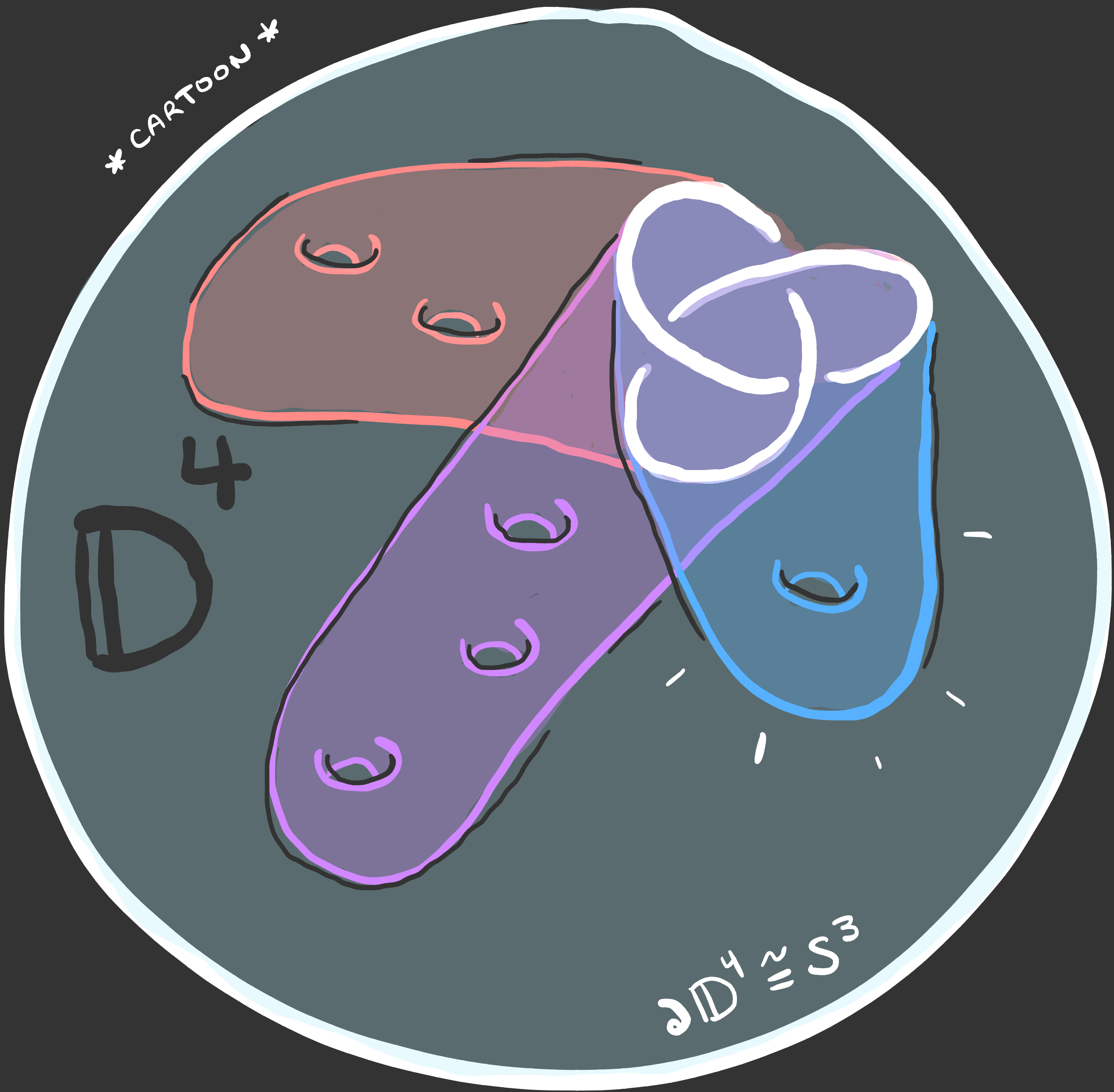
<u>Question</u>

How do you compare complexity classes of knots?

Can we measure how far they ae from being simple?

The smooth slice genus is

$$g_4(K) = \min\left\{ g(\Sigma) \;\middle|\; \begin{array}{c} \Sigma \xhookrightarrow{\text{proper}} \mathbb{D}^4 \\ \partial\Sigma = K \end{array} \right\}$$



it's hard to compute in general

A cobordism between knots K_1 & K_2 is an oriented, properly embedded surface

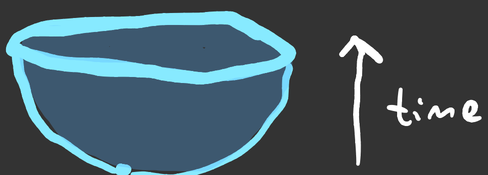
$$\Sigma \hookrightarrow S^3 \times I$$

s.t. $\partial \Sigma = K_1 \sqcup -K_2$

Morse theory

describe arbitrary cobordisms via handle attachments

$\mathbb{D}^0 \times \mathbb{D}^2$

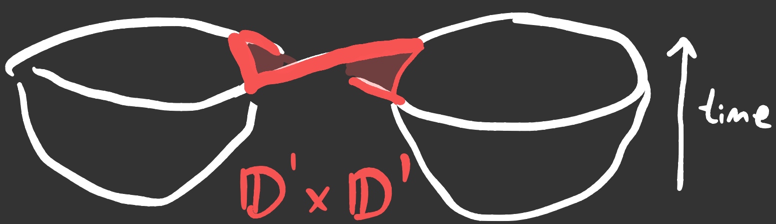


0-handle
"birth"

$\mathbb{D}^2 \times \mathbb{D}^0$

2-handle
"death"

1-handle
"band"



When $\Sigma : K_1 \longrightarrow K_2$ is an annulus, we say K_1 & K_2 are concordant.

Concordance gives an equivalence relation: $K \sim_c J$

Use to build the smooth concordance group:

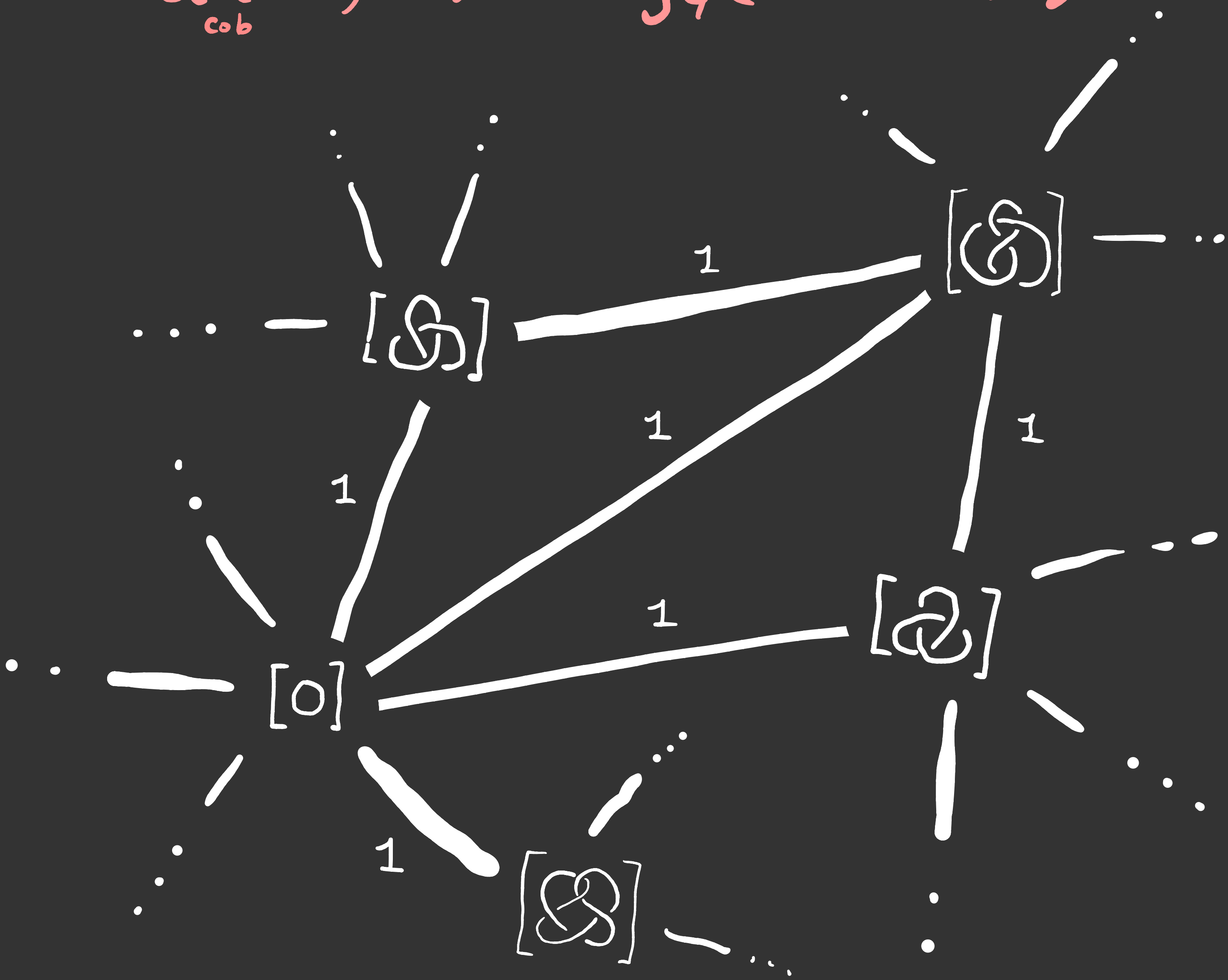
$$\mathcal{C} := \left(\{\text{Knots}\} / \sim_c \, , \; \# \right)$$

connect sum

$$\boxed{\text{Fact: } g_4(K \# J) \leq g_4(K) + g_4(J)}$$

Allows us to build a connected metric graph from $\mathcal{C}$ whose metric agrees with

$$d_{cob}(K, J) := g_4(K \# -J)$$



Today's questions

Q1 How to compute lengths of geodesics in $(\mathcal{C}, d)$?

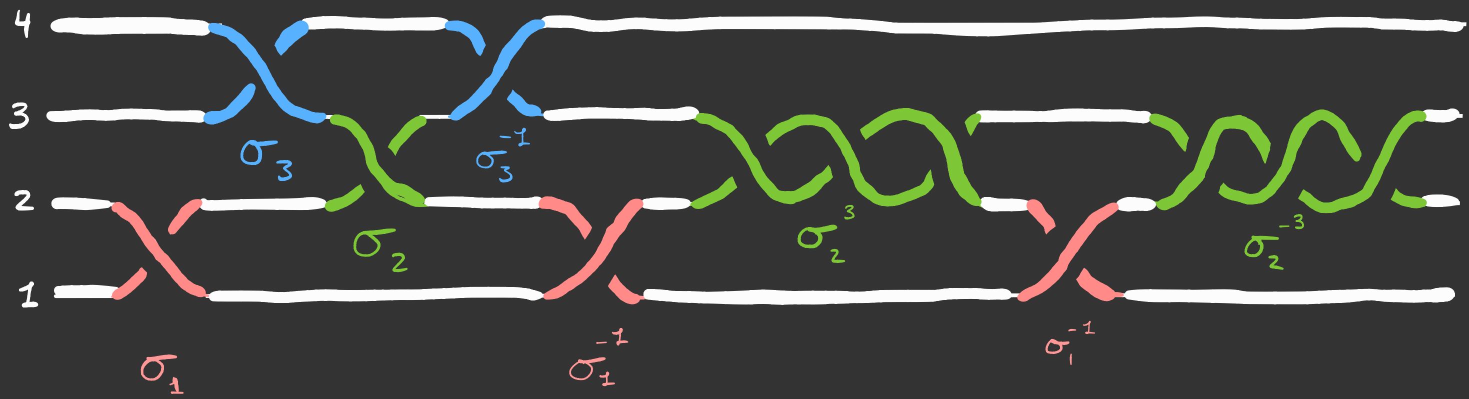
Q2 Given a geodesic Σ, what *knots lie along it?

* actually concordance classes

Q3 How does metric structure interact w/ invariants we care about?

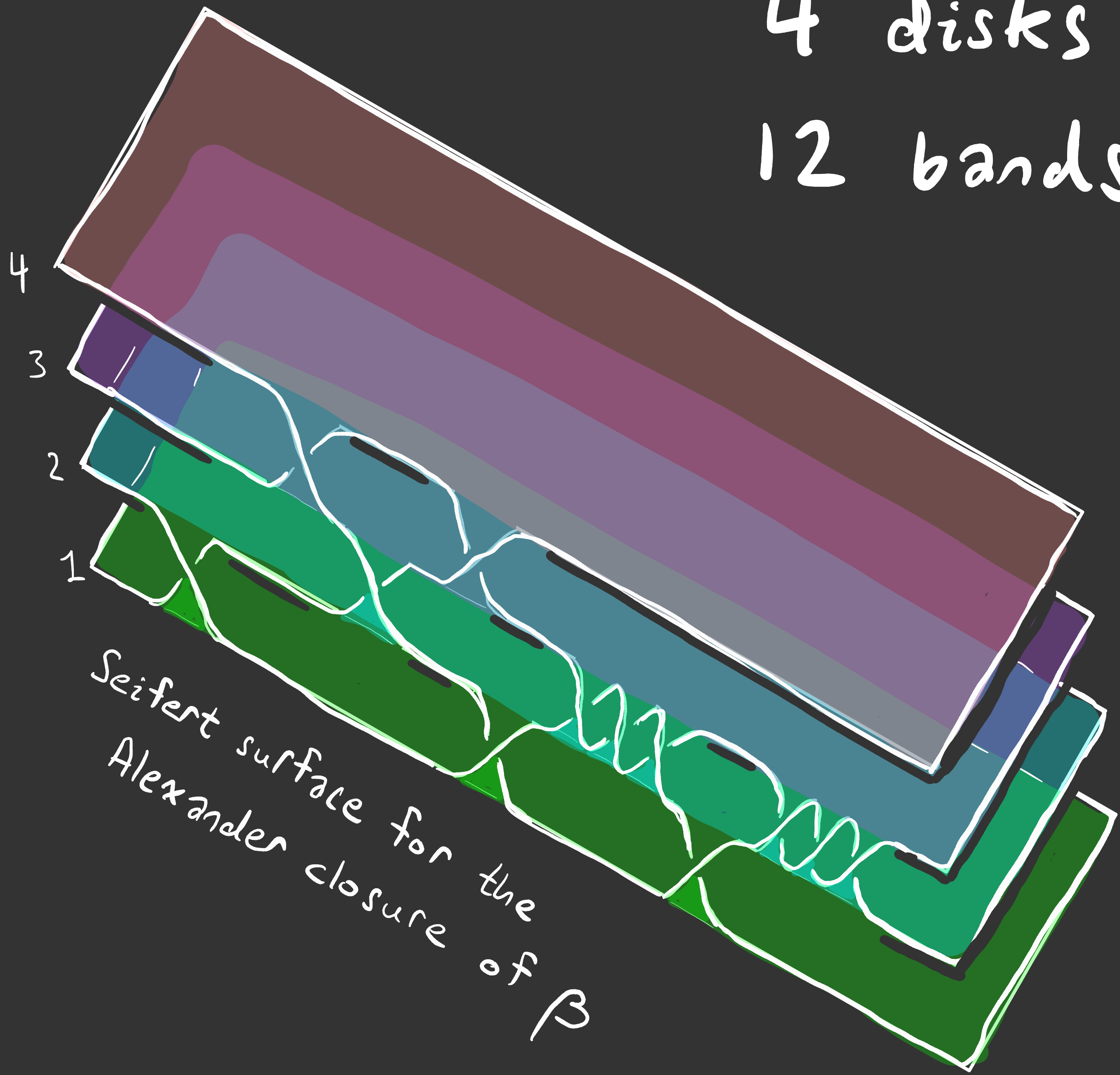
Example (Rudolph '84)

$$\beta = (\sigma_1 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_1^{-1})(\sigma_2^{3} \sigma_1^{-1} \sigma_2^{-3})$$



4 disks

12 bands

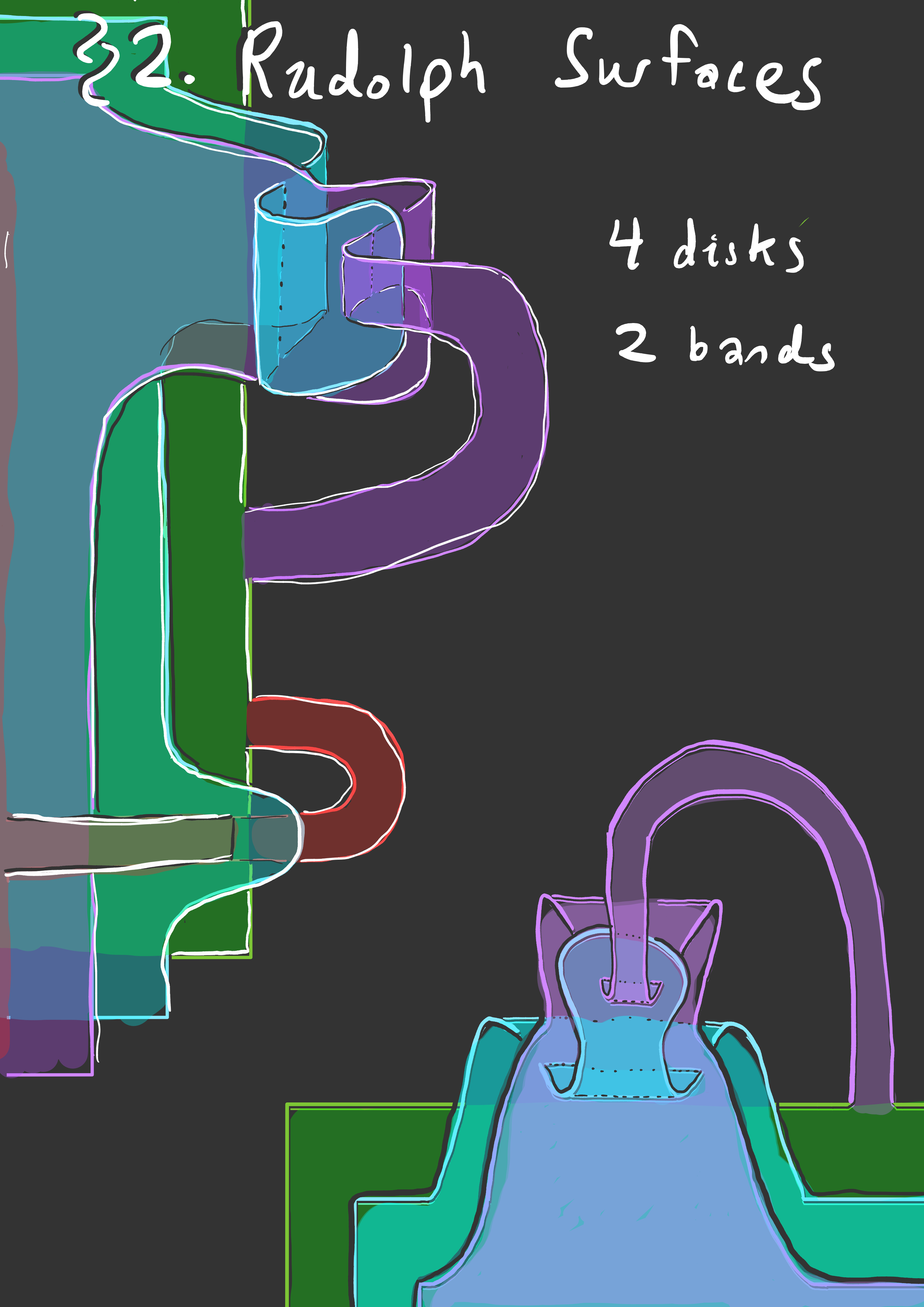


Seifert surface for the Alexander closure of β

32. Rudolph Surfaces

4 disks

2 bands



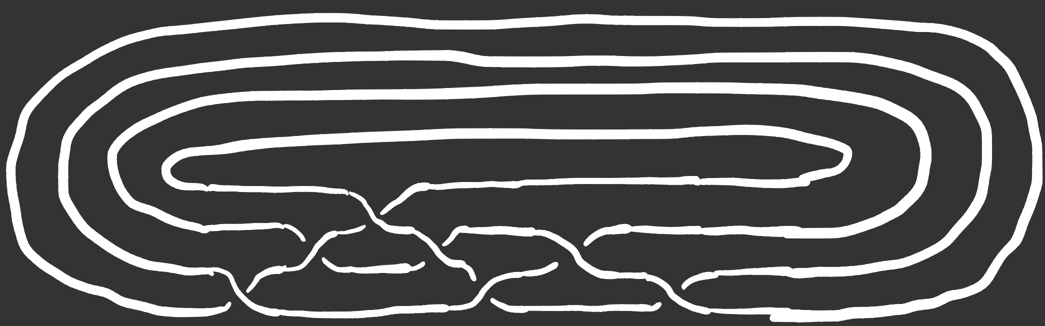
A quasipositive...

braid word is a product of conjugated Artin generators

Example: $\beta := (\sigma_1 \sigma_2^{-1} \sigma_3 \sigma_2 \sigma_1^{-1}) \sigma_2 \sigma_1$

link is one which is the closure of a quasipositive braid

$\hat{\beta}$



surface is a connected, immersed surface which is the Rudolph surface for a quasipositive link.

Prop

Quasipositive surfaces realize the slice genus of their boundary.

Proof Sketch

First, we need the

Milnor conjecture

$$g_4(T_{p,q}) = g_3(T_{p,q}) = \frac{(p-1)(q-1)}{2}$$

[Kronheimer Mrowka '93]

add Artin gens to cancel negative crossings

then add *more* until you get a torus knot

* $\exists p$ s.t. $n(p-1) - (\#\text{ pos. gens})$ more bands give $T_{n,p}$

Corollaries

- g_4 is additive for positive torus knots

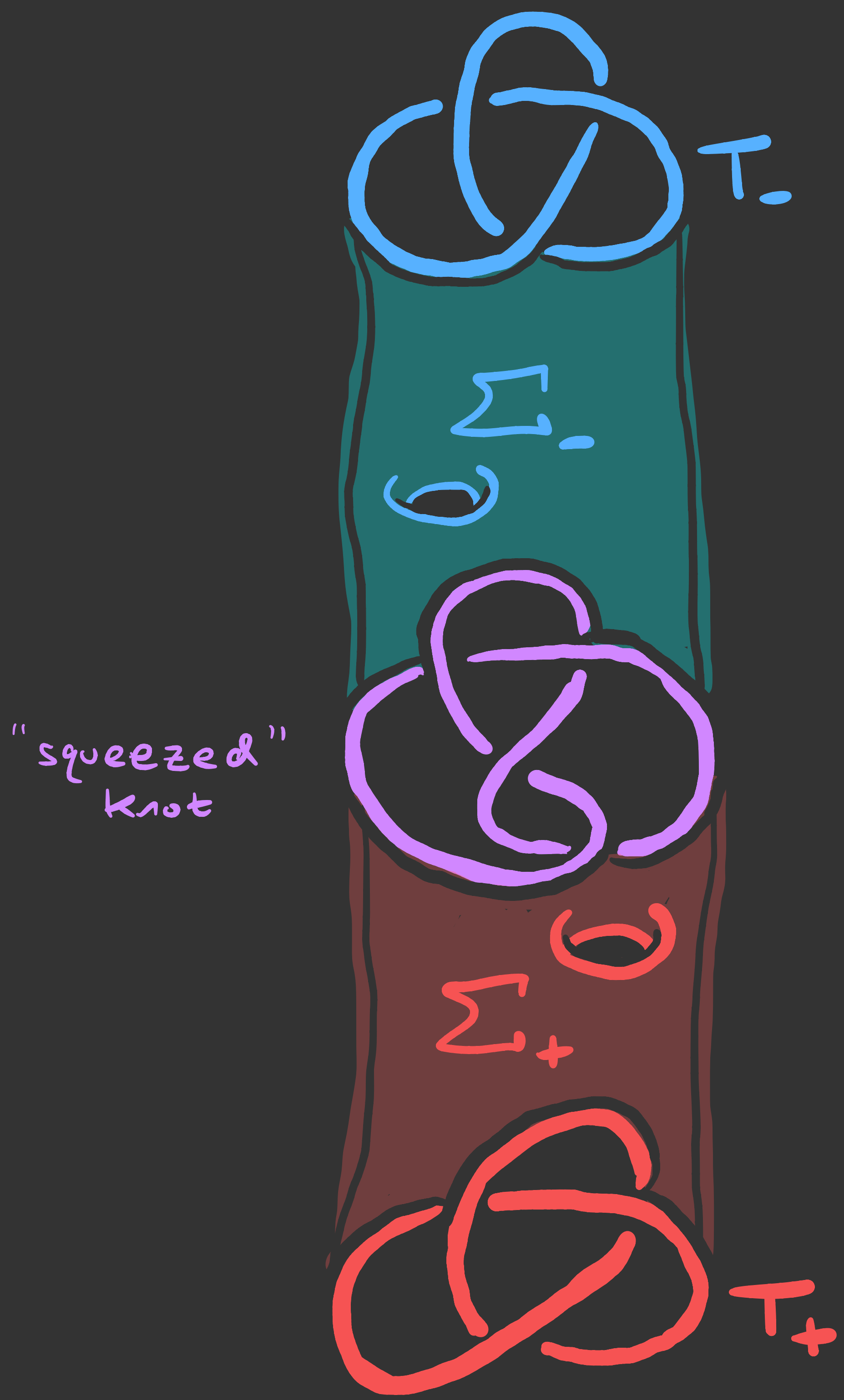
$$g_4(T_{2,3} \# T_{3,4}) = \frac{(2-1)(3-1)}{2} + \frac{(3-1)(4-1)}{2}$$

- we can compute $d(T_+, T_-)$ where T_+ is a positive torus knot, T_- negative

§3. Squeezed knots

[Feller, Lewark, Lobb '22]

DEFINITION BY CARTOON



$$\Sigma := \Sigma_+ \cup \Sigma_-$$

$$g(\Sigma) = d_{cob}(T_+, T_-)$$

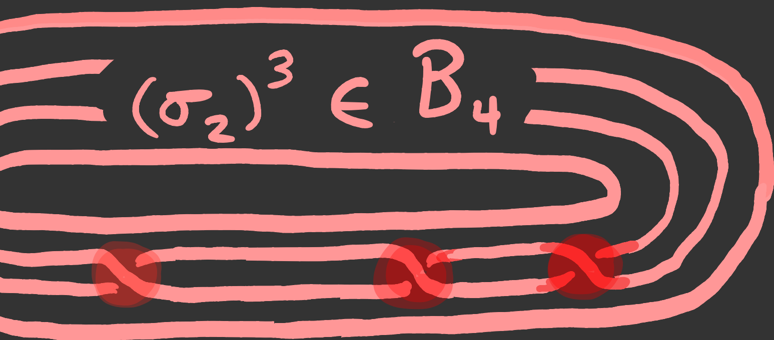
K a cross-section of Σ

Squeezed knots are knots that lie along geodesics between a positive torus knot & a negative one.

What knots are squeezed?

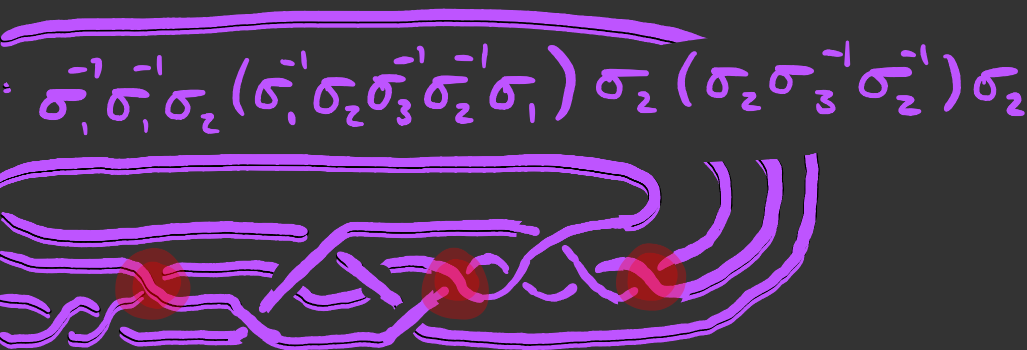
- quasipositive $\supset$ positive
- quasinegative $\supset$ negative
- alternating
- homogeneous
- quasihomogeneous [FLL '22]

$T_{2,3}$



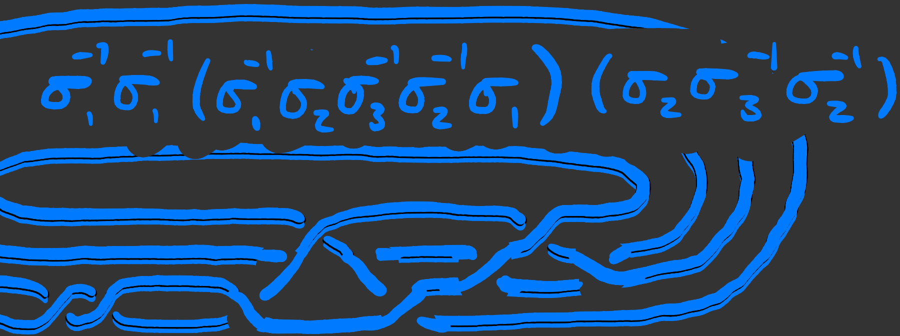
movie for
a cobordism

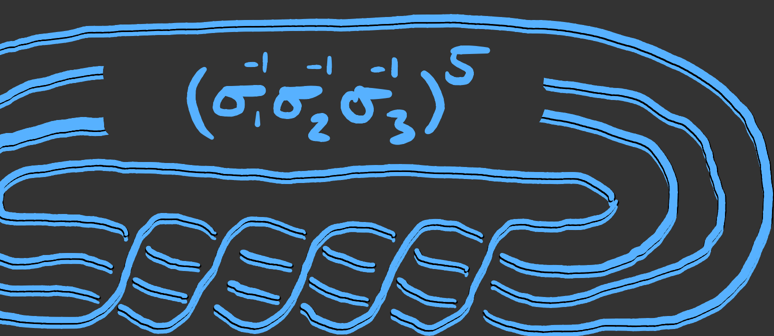
Σ



10_{146}

quasihomogenous





$-T_{4,5}$

start

$\big\downarrow$ births

$+\ 2\ \mathbb{D}^2$

$\big\downarrow$

$+ \begin{array}{l} 2\ \mathbb{D}^2 \\ 4\ \mathbb{D}^1 \times \mathbb{D}^1 \end{array}$

$\big\downarrow$

$+\ 3\ \mathbb{D}^1 \times \mathbb{D}^1$

$\big\downarrow$

$+\ 11\ \mathbb{D}^1 \times \mathbb{D}^1$

end

$$\chi(\Sigma) = 2 - 2g(\Sigma) - 2$$

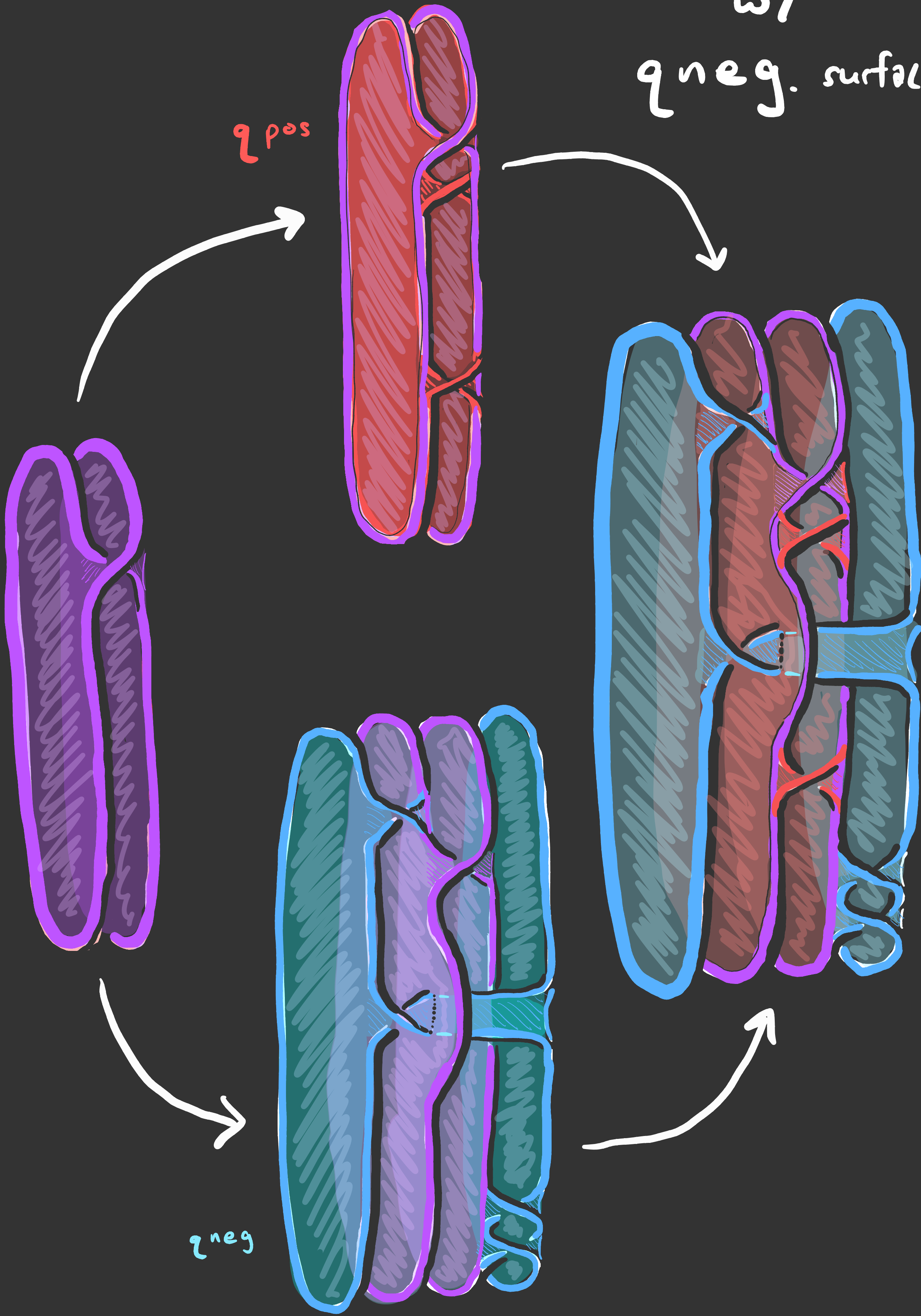
$$= \underset{\mathbb{D}^2}{4} - \underset{\mathbb{D}^1 \times \mathbb{D}^1}{18} = -14$$

$$g(\Sigma) = 7$$

$$= d_{cob}(T_{2,3}, -T_{4,5})$$

Why?

Quasihomogenous knots arise from plumbing qpos. surface w/ qneg. surface.



Remark

Of the 249 prime knots with $\leq$ 10 crossings, all but six are known to be squeezed:

9_{42}	not squeezed
10_{125}	not squeezed
10_{130}	unknown
10_{132}	not squeezed
10_{136}	not squeezed
10_{141}	unknown

How do you obstruct squeezedness?

§4 Slice torus invariants

We say a group homomorphism

$$\phi : \mathcal{C} \longrightarrow \mathbb{R}$$

is slice torus if for all knots K we have

(i) $\phi(K) \leq g_4(K)$

(ii) $|\phi(T_{p,q})| = g_4(T_{p,q})$

Notation:

$$ST = \left\{ \begin{array}{c} \text{slice torus} \\ \text{invariants} \end{array} \right\}$$

Examples

from Khovanov homology

- $s/2$ (Rasmussen, 2003)

from Heegaard Floer

- (Oszváth, Szabó 2003, 2011)

  $\tau, \nu : \mathcal{C} \longrightarrow \mathbb{R}$

from Framed Instanton Floer

- (Baldwin, Sivek 2020)

  $\tau^{\#}, \nu^{\#} : \mathcal{C} \longrightarrow \mathbb{R}$

Stable Knot Invariants

For K a knot, its

stable 4-genus is

$$\hat{g}_4(K) := \lim_{n\to\infty} \frac{g_4(\#^n K)}{n}$$

[Livingston '10]

&

its "**slice torus bound**" is

$$\underbrace{\ell(K) := \lim_{p\to\infty} \hat{g}_4(T_{p,p+1} \# K) - \hat{g}_4(T_{p,p+1})}$$

[FLL '22]

Observations

- $\hat{g}_4(K) \leq g_4(K)$

Pf $\quad g_4(K) = \dfrac{n\, g_4(K)}{n} \geq \dfrac{g_4(nK)}{n}$

- $\hat{g}_4(T_{p,q}) = g_4(T_{p,q})$

Pf g_4 additive for positive torus knots, run argument above.

Theorems [FLL '22]

(i) If K is squeezed then

$$\ell(K) = -\ell(-K)$$

(ii) Let K be a knot, then

$$[-\ell(-K), \ell(K)] = \{\phi(K) \mid \phi \in ST\} \subset \mathbb{R}$$

Corollary

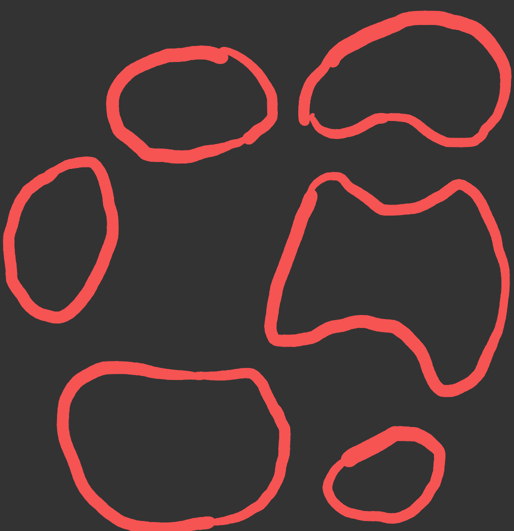
If K is squeezed, then for all slice torus ϕ, ψ we have

$$\phi(K) = \psi(K).$$

§4 Turaev Genus

Resolutions

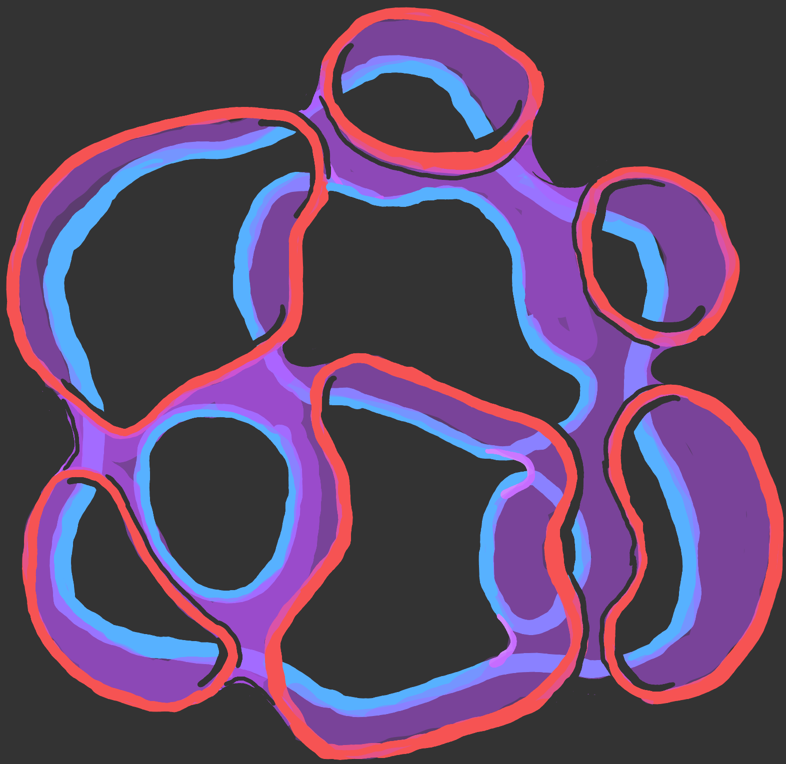
8_8



$s_A(D) = 6$ $\qquad$ $s_B(D) = 4$

connect the components at crossings w/ saddle cobordisms; get a Turaev surface of genus

$$g_T(D) = \frac{c(D) + 2 - |s_A| - |s_B|}{2}$$



$$\frac{8 + 2 - 6 - 4}{2} = 0$$

Turaev genus

$$g_T(L) := \min\left\{ g(\Sigma) \;\middle|\; \begin{array}{c} \Sigma \text{ a Turaev} \\ \text{surface for} \\ L \end{array} \right\}$$

- $g_T(L) = 0$ iff L alternating
  [Turaev '87]

- used to give a simpler proof of the Tait conjecture
  [Turaev '87]

- bounds width of reduced Khovanov homology / Knot Floer homology

$$\omega_{\widetilde{Kh} \text{ or } HFK}(K) \leq 1 + g_T(K)$$

[Champanerkar, Kofman '09]
[Lowrance, '08]

Results of Jung, Kang, Kim

Theorem (JKK, '22)

If $\phi \in ST$, can build a Dasbach-Lowrance invariant:

$$\tilde{\phi}(L) := 2\phi(L) - (n-1)$$

Theorem (JKK, '22)

If α, β are Dasbach-Lowrance invariants, then

$$\frac{1}{2}\left|\alpha(K) - \beta(K)\right| \leq g_T(K)$$

New bound on Turaev genus!

<u>Corollary</u> [FLL & JKK]

$$\ell(K) + \ell(-K) \leq g_T(K)$$

Open Questions

1. Are 10_{130} & 10_{141} squeezed?

2. Does $\ell(K) + \ell(-K) = 0$ force squeezedness?

3. Can this story be extended to torus links?

4. Does squeezedness have implications for corresponding 3-manifolds?

Thank you for listening!

Special thanks to

- Akram Alishahi
- Han Lou
- Sasha Shmakov

References

Champanerkar & Kofman

① A Survey on the Turaev genus '14

Feller, Lewark, & Lobb

② Squeezed Knots '22

③ On Values taken on by slice torus invariants '22

Jung, Kang, & Kim

④ Concordance invariants and the Turaev genus '22

Rudolph

⑤ Braided Surfaces and Seifert Ribbons for closed braids '83