I. The Milnor Conjecture
II. Squeezed knots
III. Slice torus invariants and squeezed knots
IV. Obstructions
V. Future directions

Squeezed Knots & Slice Torus Invariants

Gary D Dunkerley

Oral Examination

November 15th 2023

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- Khovanov homology
- Lee homology
- The s-invariant
- Smooth 4-genus bound
- Proof of Milnor Conjecture

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In this presentation, everything is smooth.

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The Milnor Conjecture

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Theorem (Kronheimer-Mrowka '93)

$$\underbrace{g_4(T_{p,q})}_{\substack{smooth\\4-genus}} = \underbrace{g_3(T_{p,q})}_{\substack{Seifert\\genus}} = \frac{(|p|-1)(|q|-1)}{2}$$

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We will present a simpler proof of the above first noticed by Jake Rasmussen.

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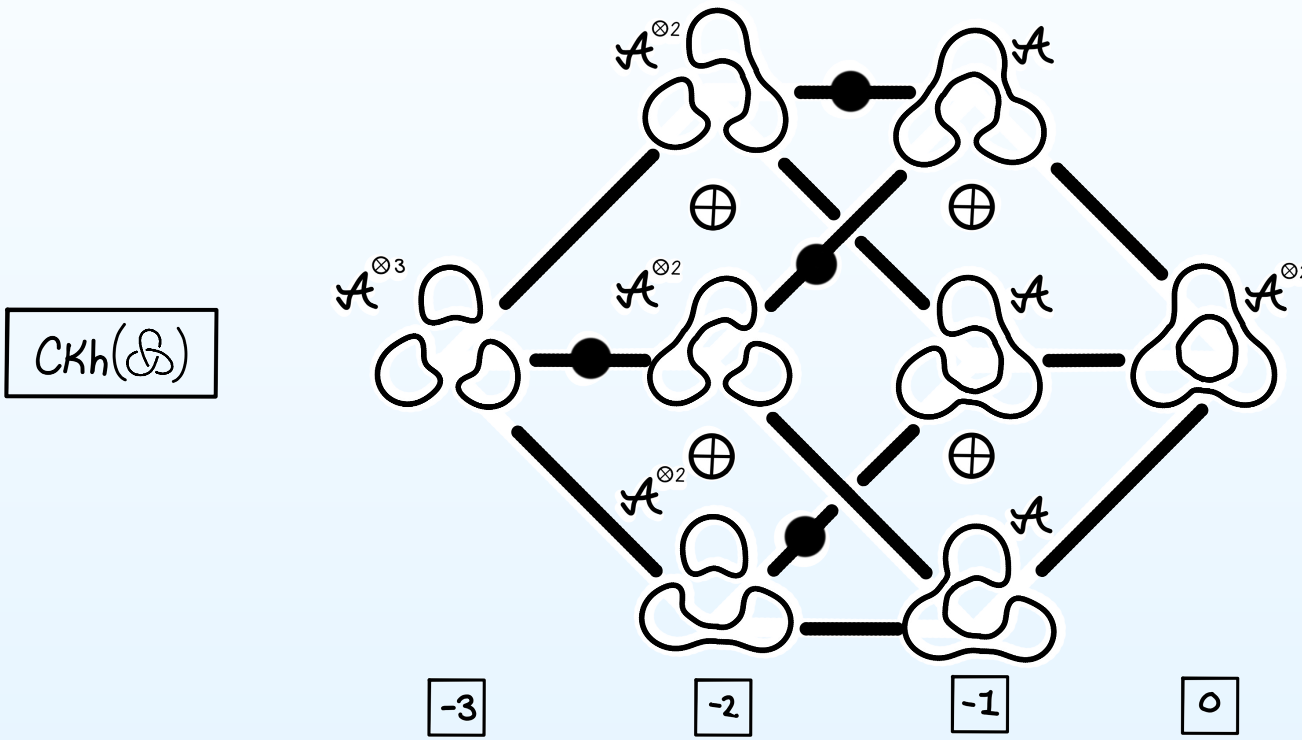
Khovanov homology

Khovanov defined a bigraded (co)homology for knots using the **cube of resolutions** of a diagram D.



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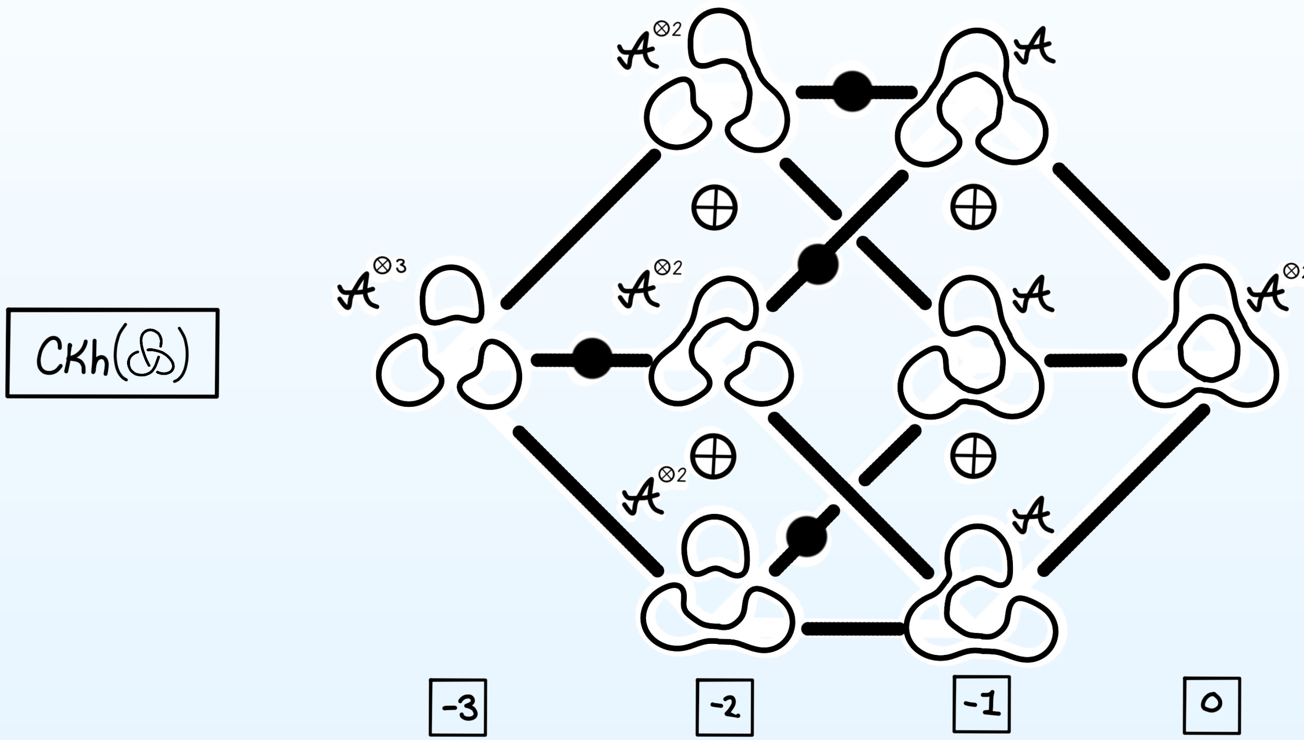

Generators are resolutions with components labeled by generators $\{v_\pm\}$ of the Frobenius algebra $\mathcal{A} \cong \frac{\mathbb{F}[x]}{(x^2)}$. Differential is defined using the multiplication and comultiplication of $\mathcal{A}$.

Gradings: (co)homological and quantum

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Let N be the number of crossings in D and let $\epsilon \in \{0,1\}^N$ encode a resolution D_ϵ with C components.

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Let $\alpha = v_1 \otimes \cdots \otimes v_C \in \mathcal{A}^{\otimes C} =: \mathcal{A}(D_\epsilon)$.

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$$|\epsilon| := \sum_{i=1}^{n} \epsilon(i) \qquad N_\pm := \#(\pm\text{-crossings})$$

$$\deg(v_{\pm 1}) = \pm 1 \qquad \deg(\otimes^i v_i) := \sum \deg(v_i)$$

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(co)homological grading

$$gr_h(\alpha) := |\epsilon| - N_-$$

quantum grading

$$gr_q(\alpha) := \deg(\alpha) + gr_h(\alpha) + N_+ - N_-$$

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The q-grading on Khovanov homology defines a (descending) filtration of $\mathrm{CKh}(K;\mathbb{F})$:

$$\mathcal{F}_j = \{\alpha \in \mathrm{CKh}(K;\mathbb{F}) \mid gr_q(\alpha) \geq j\}$$

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$$\mathrm{Kh}_{Lee}(K) = \bigoplus_{h,s\in\mathbb{Z}} \mathrm{Kh}^{h,s}_{Lee}(K)$$

where the s-grading is induced by the filtration $\mathcal{F}_\bullet$.

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where the s-grading is induced by the filtration $\mathcal{F}_\bullet$.
For Σ a connected, oriented cobordism between knots K_0 and K_1, there's a filtered isomorphism $\phi_S : \mathrm{Kh}_{Lee}(K_0) \to \mathrm{Kh}_{Lee}(K_1)$ of degree $-2g(\Sigma)$.

Lee generators

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For L a link on n-components, Lee also proved $\mathrm{Kh}_{Lee}(L) \cong \mathbb{Q}^{2n}$ via a concrete bijection of generators and orientations on L. We will use the basis

$$a = v_- + v_+ \qquad b = v_- - v_+$$

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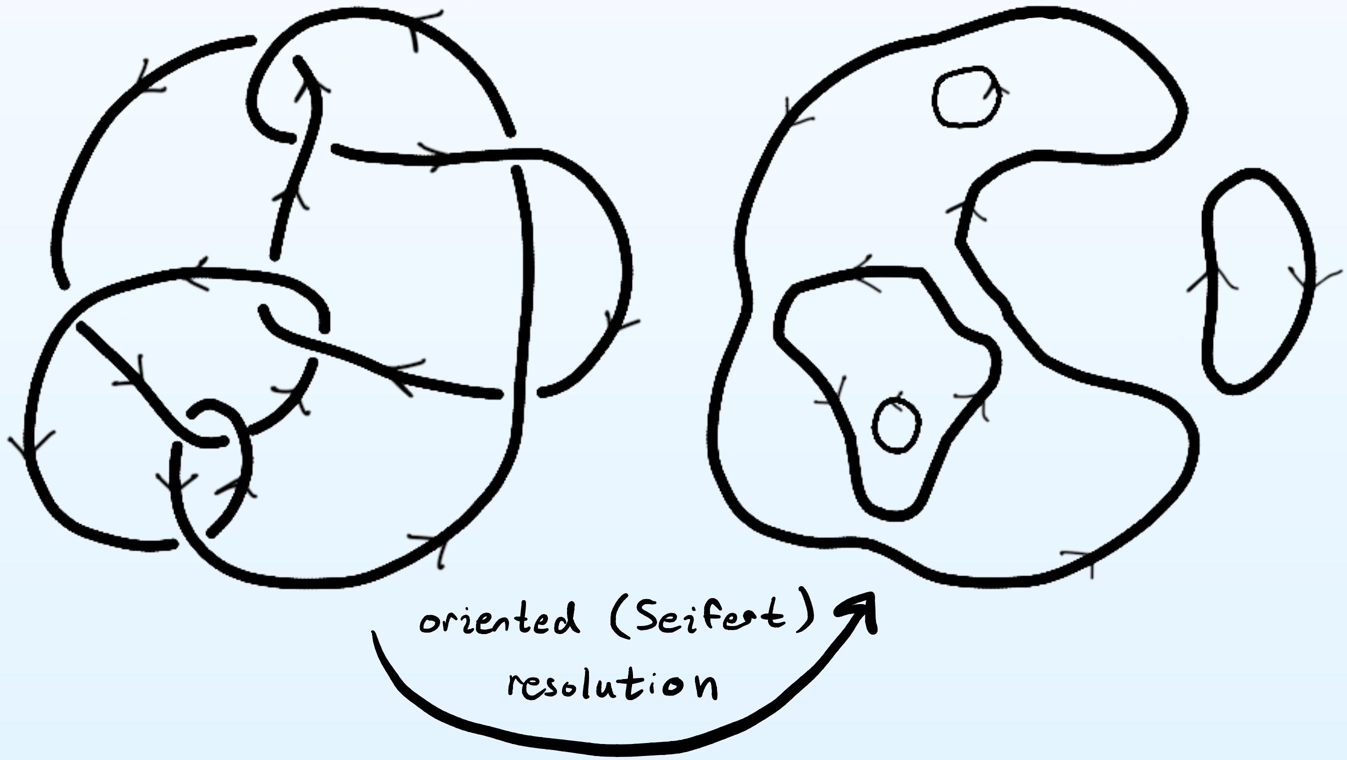

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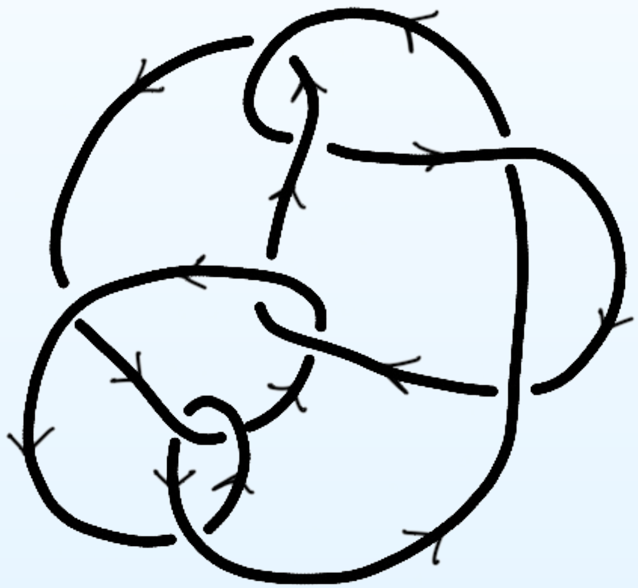

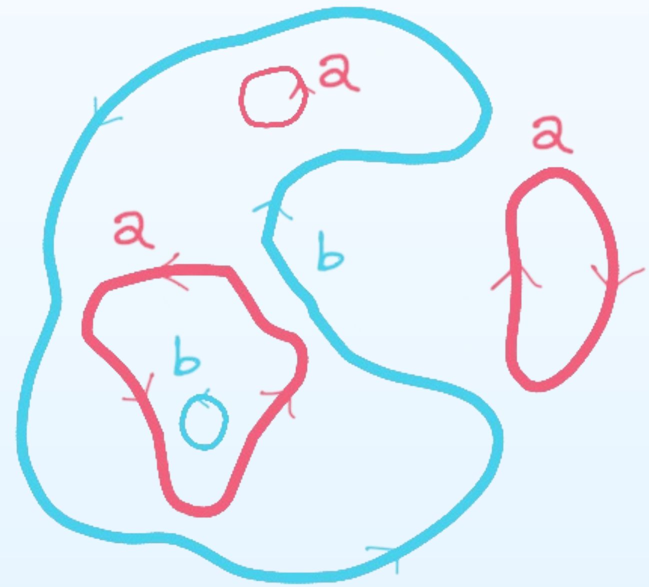


```
if ( count(C)  + isCounterClockwise(C) = 0 mod 2 ):
        label(C) = a
else:
        label(C) = b
```

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- For a knot K, Rasmussen noticed the largest s grading supported in $\mathrm{Kh}_{Lee}(K)$ is always 2 more than the smallest supported s-grading.

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- For a knot K, Rasmussen noticed the largest s grading supported in $\mathrm{Kh}_{Lee}(K)$ is always 2 more than the smallest supported s-grading.

$$s(K) := s_{\max}(K) - 1 = s_{\min}(K) + 1$$

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- With these, he was able to show $|s(K)| \leq 2g_4(K)$ and in particular $|s(T_{p,q})| = 2g_4(T_{p,q})$. We offer the proof.

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$$2g_4(K) + 1 \geq s(\alpha) = s_{max}(K)$$

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$$2g_4(K) \geq s(\alpha) - 1 = s(K)$$

Let S_- be the mirror of S_+: this gives a cobordism from the concordance inverse $-K$ to the unknot U. By same argument get

$$s(-K) \leq 2g_4(K)$$

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Therefore $|s(K)| \leq 2g_4(K)$.

$\square$

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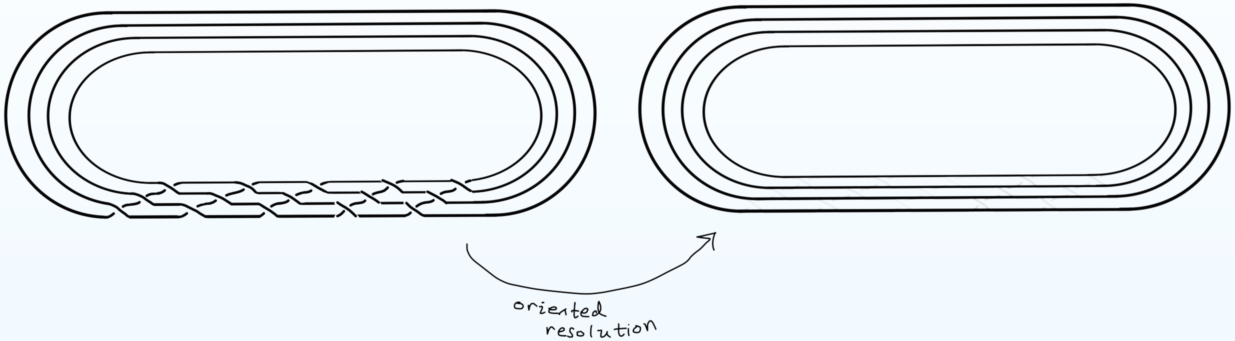

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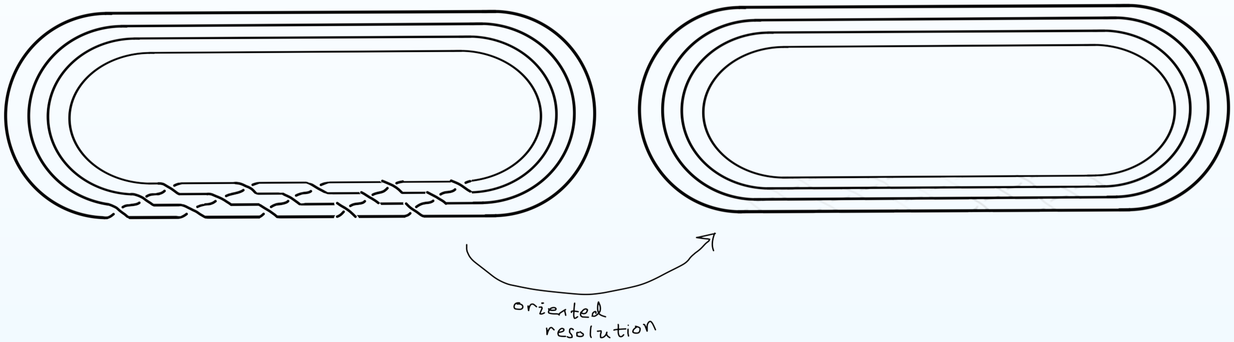

note for any orientation o, we have

$$gr_h(\mathfrak{s}_o) = 0$$

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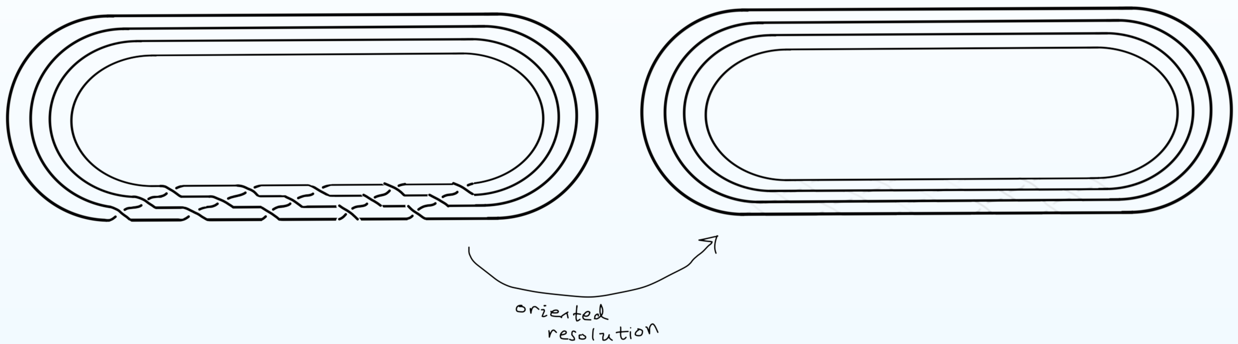

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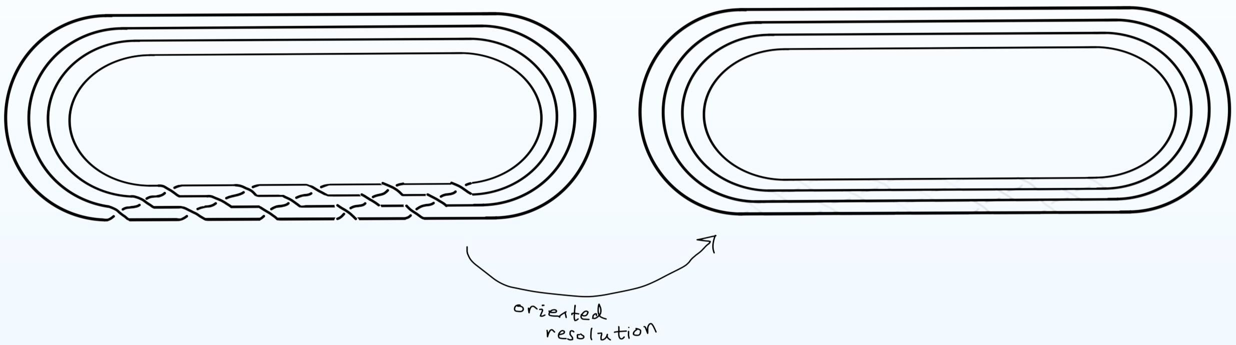

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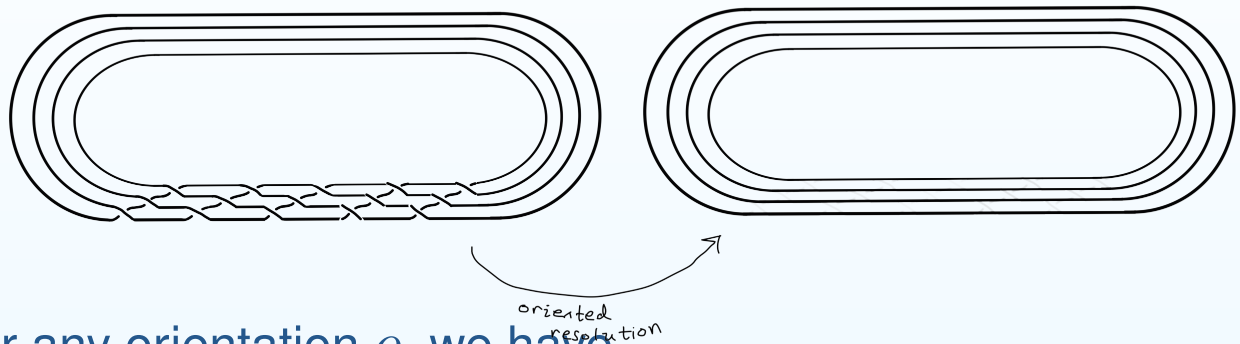

note for any orientation o, we have

$$\underbrace{gr_h(\mathfrak{s}_o) = 0}_{\text{(lowest possible)}} \xrightarrow{[\alpha]=[\mathfrak{s}_o] \Leftrightarrow \alpha=\mathfrak{s}_o} s[\mathfrak{s}_0] = \text{smallest q-grading of summand in } \mathfrak{s}_o$$

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- Khovanov homology
- Lee homology
- The s-invariant
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II. Squeezed knots

III. Slice torus invariants and squeezed knots

IV. Obstructions

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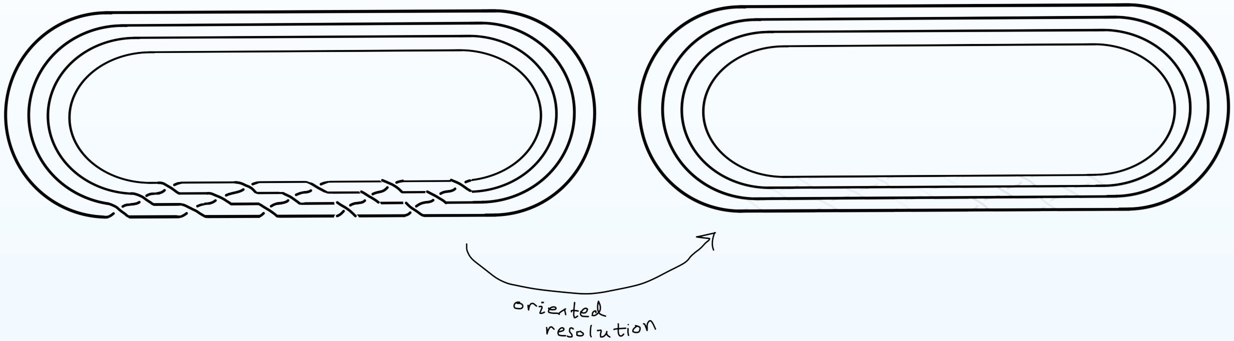

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$$= \underbrace{-(\#\text{components})}_{\deg(v_-\otimes\cdots\otimes v_-)} + \underbrace{0}_{gr_h(\mathfrak{s}_{\vec{0}})} + N_+ - \underbrace{0}_{N_-}$$

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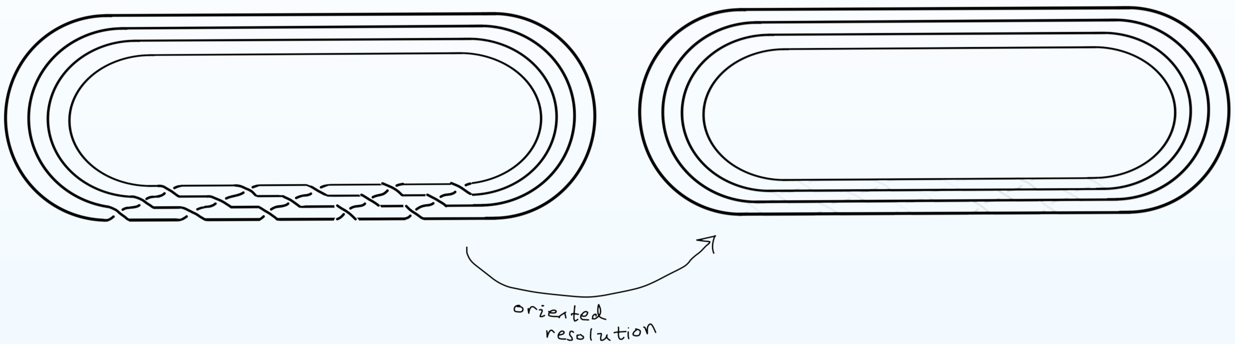

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Observe diagram gives Seifert surface with

$$\chi(\Sigma) = -s(\mathfrak{s}_o) = -s_{\min}(K)$$

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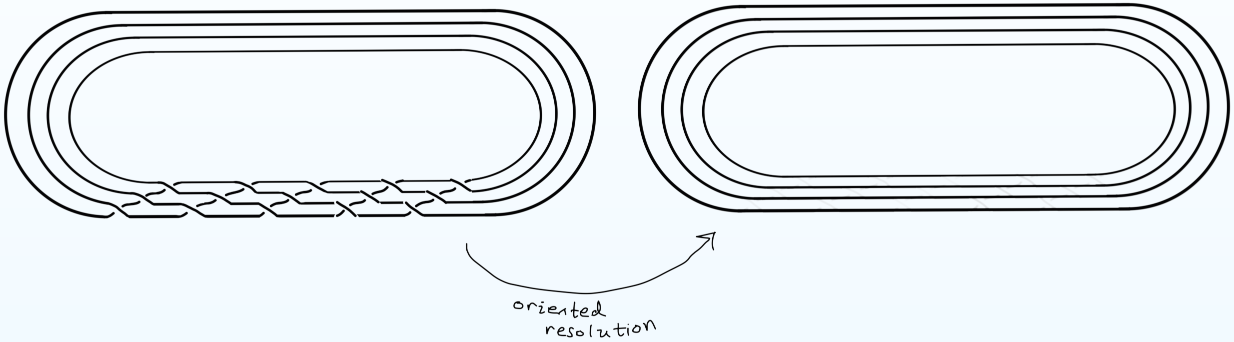

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Observe diagram gives Seifert surface with

$$\chi(\Sigma) = -s(\mathfrak{s}_o) = -s_{\min}(K) \Rightarrow -\chi(\Sigma) + 1 = s(K)$$

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Finally, conclude

$$2g_3(K)$$

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$$2g_3(K) \leq 2g(\Sigma)$$

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Finally, conclude

$$2g_3(K) \leq \underbrace{2g(\Sigma) = 2 - \chi(\Sigma) - 1}_{\text{genus formula}}$$

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$$2g_3(K) \leq \underbrace{2g(\Sigma) = 2 - \chi(\Sigma) - 1}_{\text{genus formula}} = s(K) \leq 2g_4(K)$$

Since $g_3(K) \geq g_4(K)$ in general, have equality.

$\square$

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The cobordism distance

Let K and J be knots. Define their **cobordism distance** to be

$$d_{cob}(K, J) := g_4(K\# - J)$$

The cobordism distance

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Squeezed knots are cross-sections of genus minimizing cobordisms between a positive torus knot and a negative torus knot.

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Species of squeezed knots

Other than torus knots, many natural and interesting classes are squeezed!

- **Positive knots**
  Knots that admit diagrams with only positive / negative crossings.
- **Alternating knots**
  Knots that admits a diagram whose crossings alternate between over and under.
- **Quasipositive knots** Knots arising as closures of braid words consisting of conjugated Artin generators.
- **Quasihomogenous knots** Knots arising from an ribbon-immersed plumbing of two surfaces together along a disk: one bounded by a quasipositive knot and the other bounded by a quasinegative knot.

Quasipositive knots

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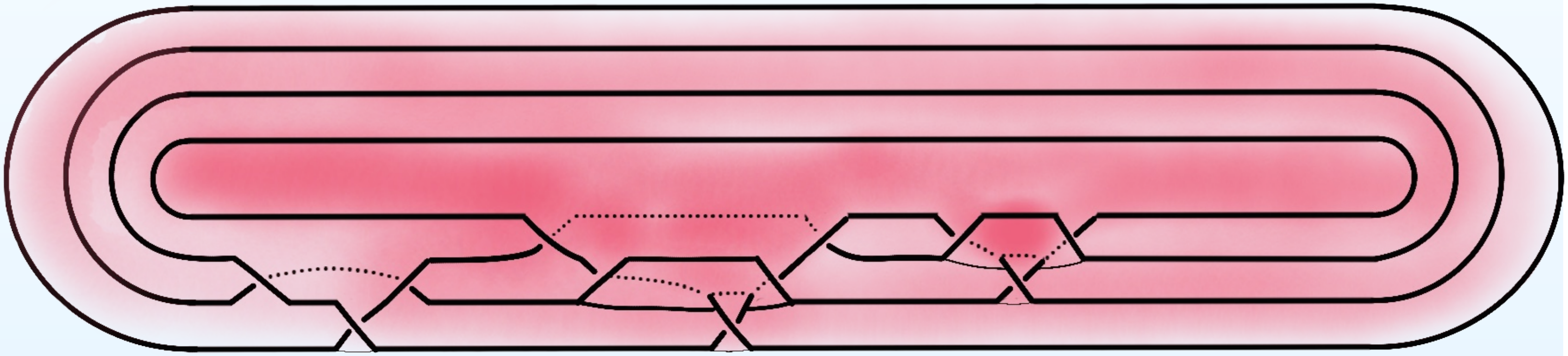
- Choose a quasipositive braid β, the conjugated Artin generators in β give ribbon-immersed bands between the Seifert circles of $\overline{\beta}$. Result is the **Rudolph surface** of $\overline{\beta}$.



- Puncturing one of the Seifert disks of $\overline{\beta}$ turns R into a cobordism from $\overline{\beta}$ to the unknot.

Quasipositive knots are squeezed

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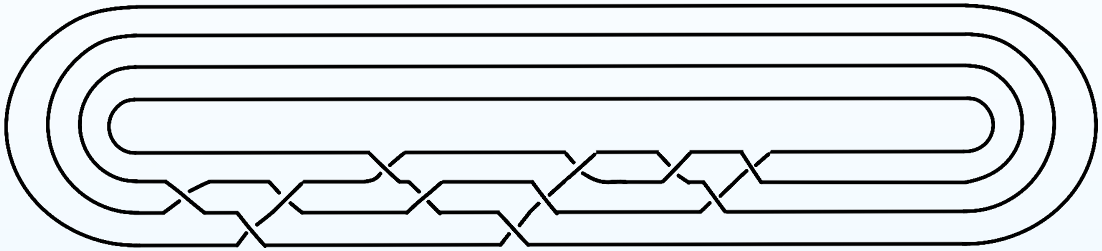
- Let $\beta \in B_n$ be a quasipositive braid word.



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Quasipositive knots are squeezed

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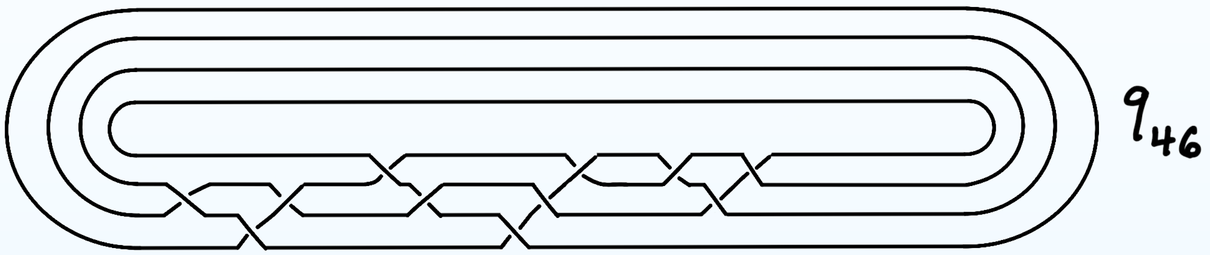
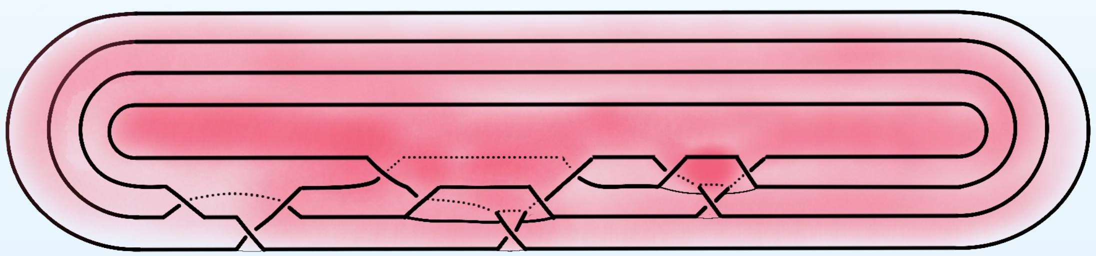


- Suppose β has I inverse Artin generators and $A + I$ Artin generators and let R be its associated (once-punctured) Rudolph surface.



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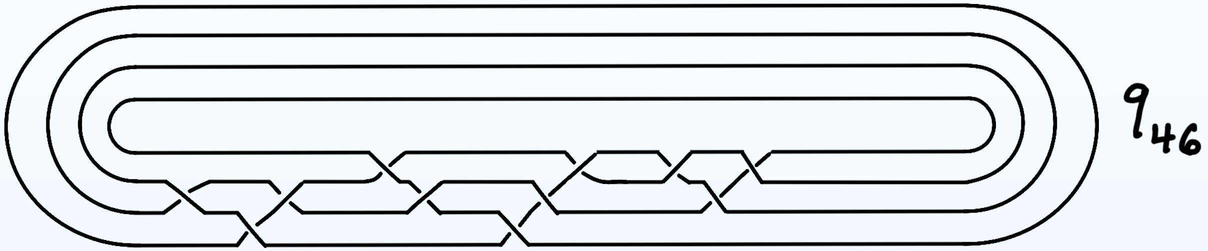

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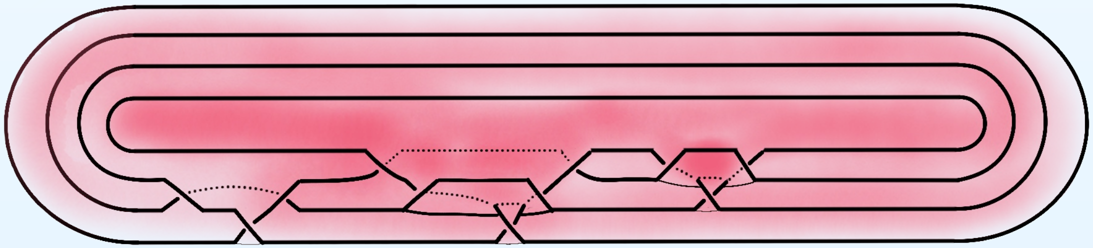

- Goal: extend R to get a cobordism Σ from a torus knot to the unknot and then show that cobordism has minimal genus.

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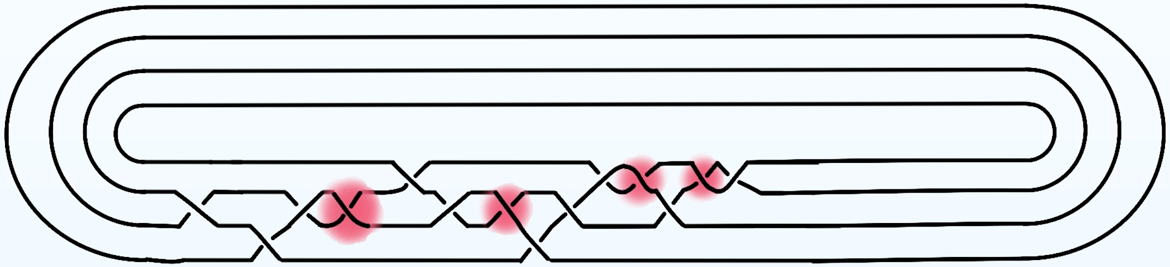
Quasipositive knots are squeezed

- Add I positive bands to the braid to cancel the inverse Artin generators.



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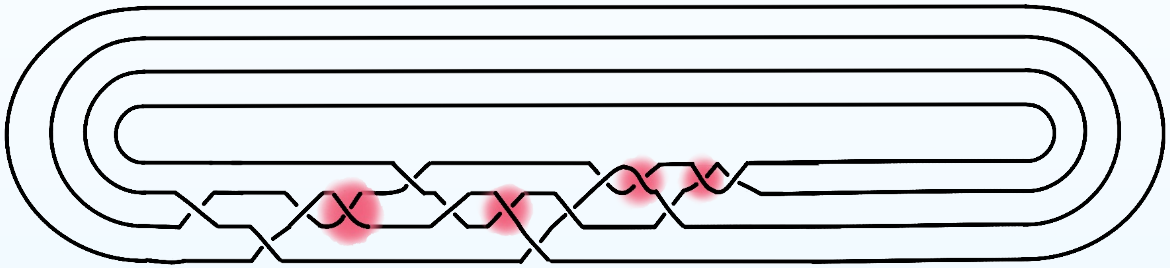
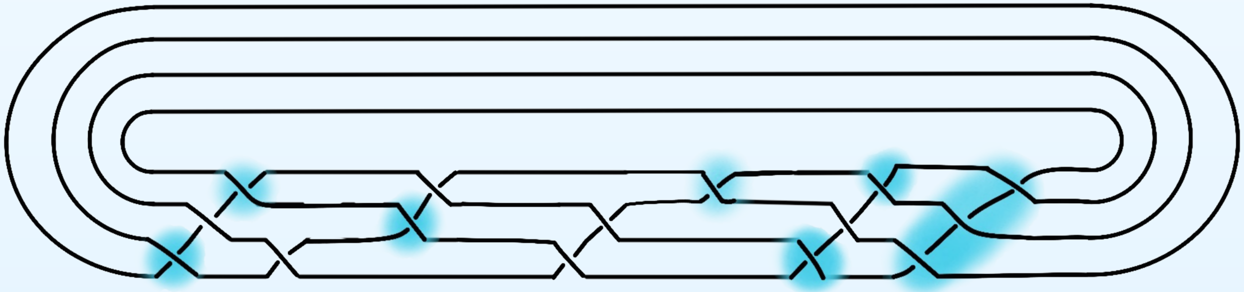


- For some p, we can add $p(n-1)-(A+I)$ additional positive bands to get a braid whose closure is $T_{n,p}$.



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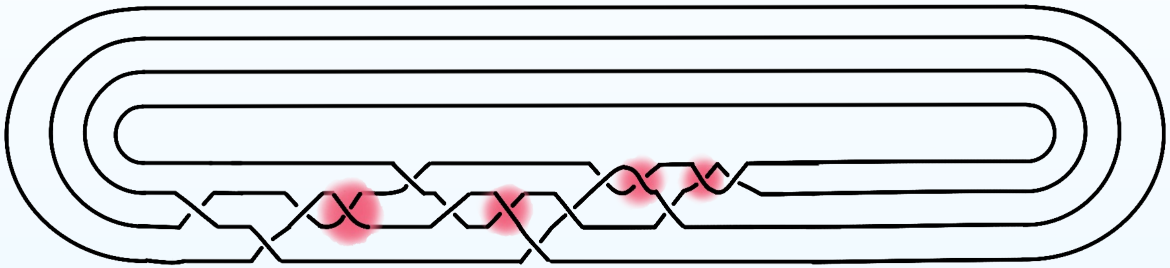

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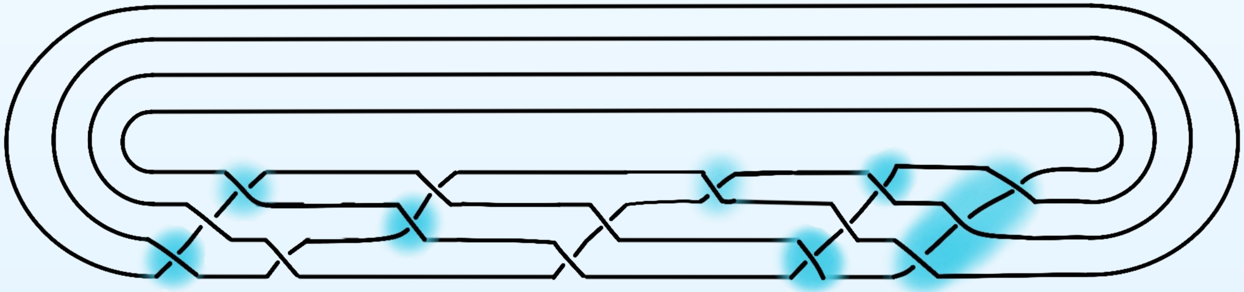

- Adding these bands to $\overline{\beta}$ gives a cobordism C from $\overline{\beta}$ to $T_{n,p}$.

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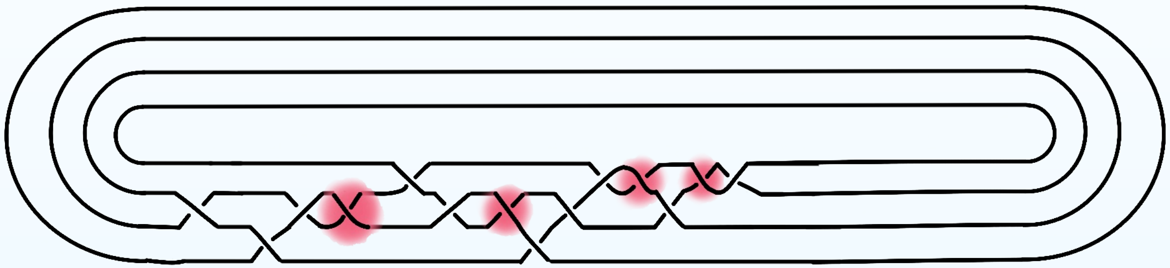

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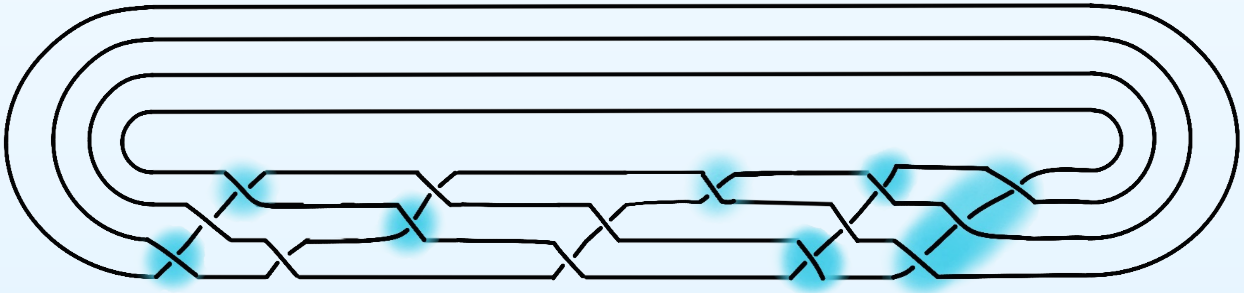

- Adding these bands to $\overline{\beta}$ gives a cobordism C from $\overline{\beta}$ to $T_{n,p}$. Concatenating C with R gives a cobordism Σ from $T_{n,p}$ to the unknot.

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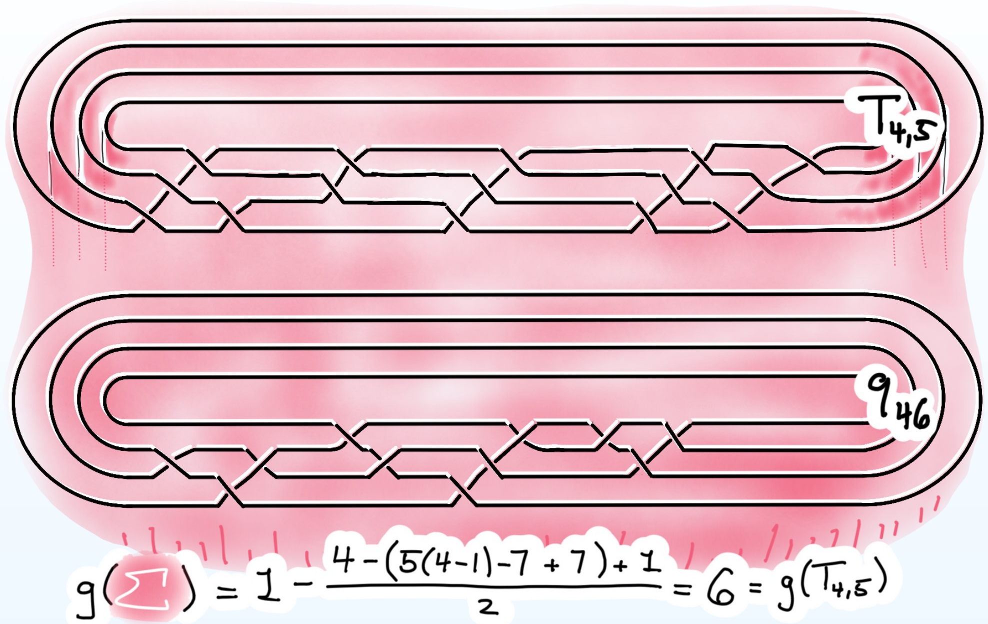
Quasipositive knots are squeezed



Note the Euler characteristic of Σ is

$$n - \Big((p(n-1) - (A+I)) + (A+I) \Big) = n - pn + p$$

From which the genus formula yields

$$g(\Sigma) = 1 - \frac{(n-pn+p)+1}{2} = \frac{(n-1)(p-1)}{2} = g_4(T_{n,p})$$

Therefore Σ is genus minimizing. $\square$

Consequences

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Corollary

If R is a Rudolph surface for a quasipositive knot K, then $g_4(K) = g(R)$.

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Corollary

g_4 is additive on quasipositive knots (and therefore, on positive torus knots).

Squeezing with qpos. and qneg. knots

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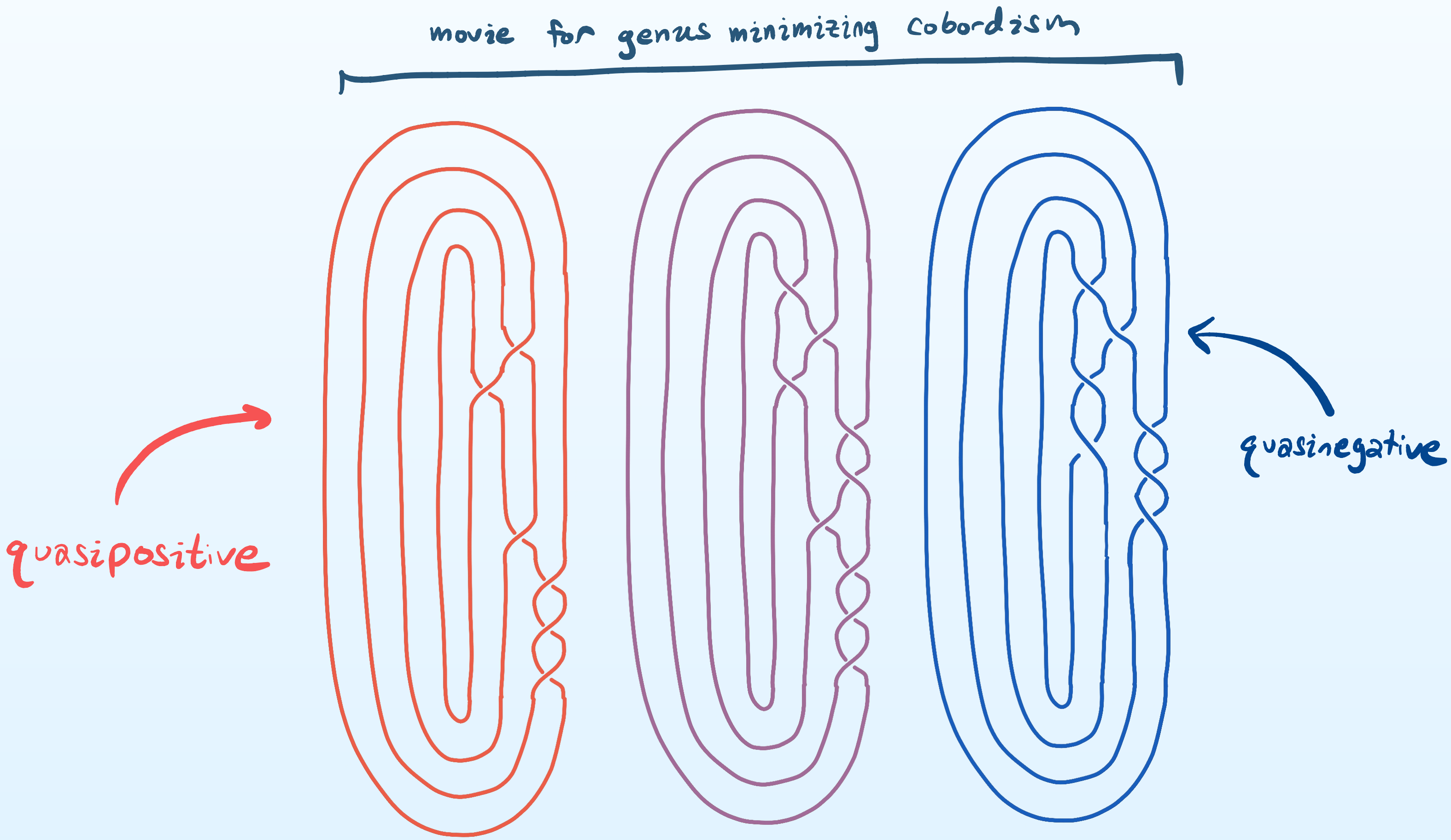
Lemma

If K is a slice of a (connected) genus minimizing cobordism from a (quasi)positive knot to a (quasi)negative knot, then K is squeezed.



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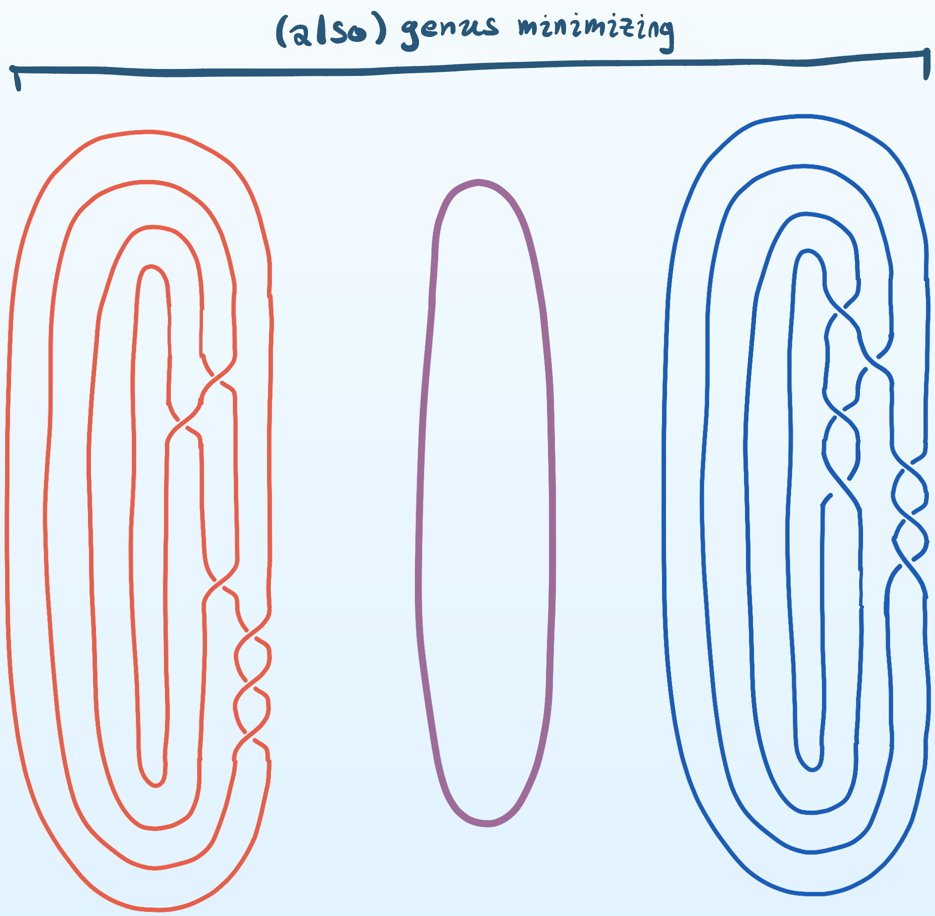

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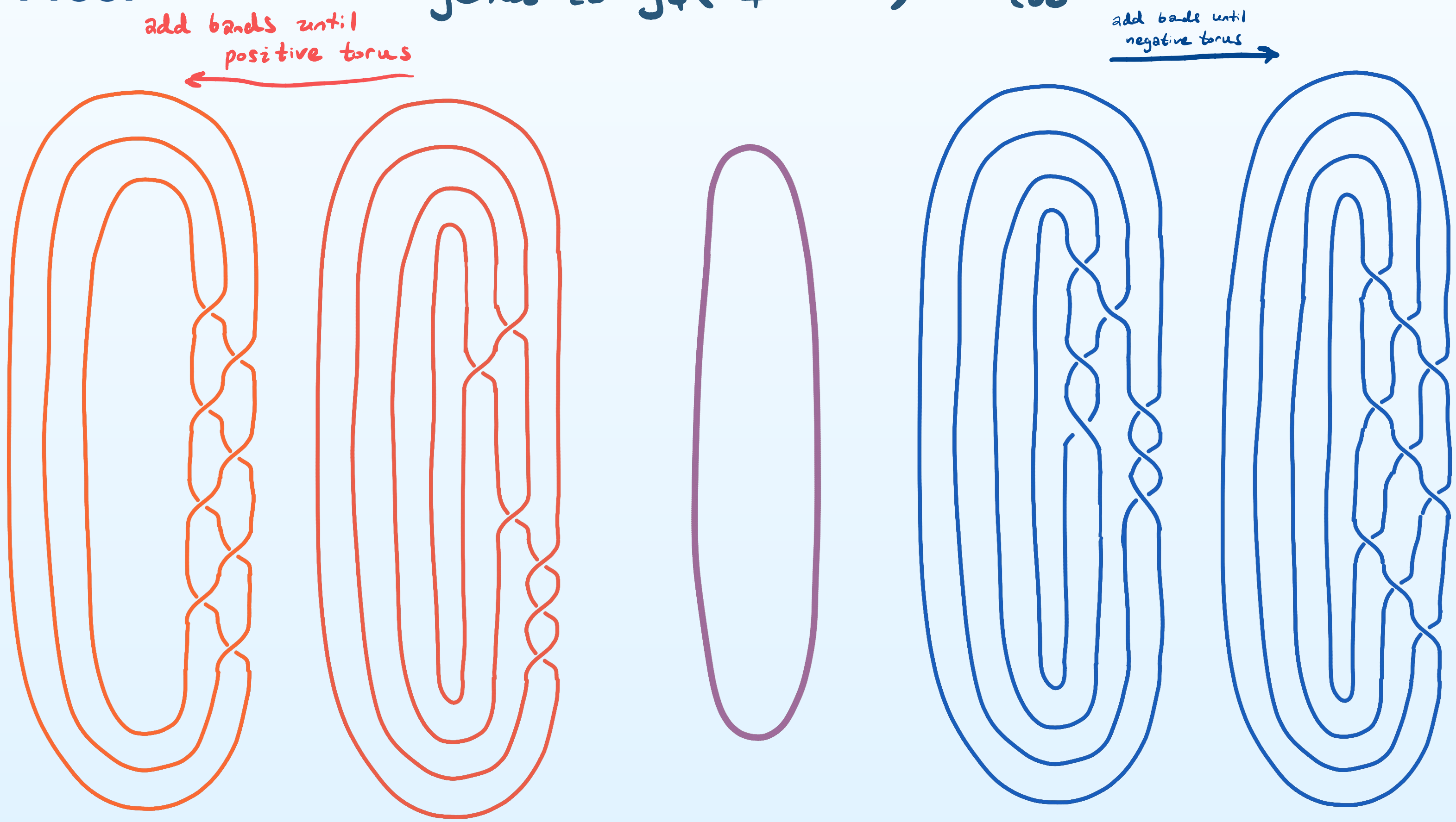
genus is $g_4(T_+ \# -T_-) = d_{cob}(T_+, T_-)$



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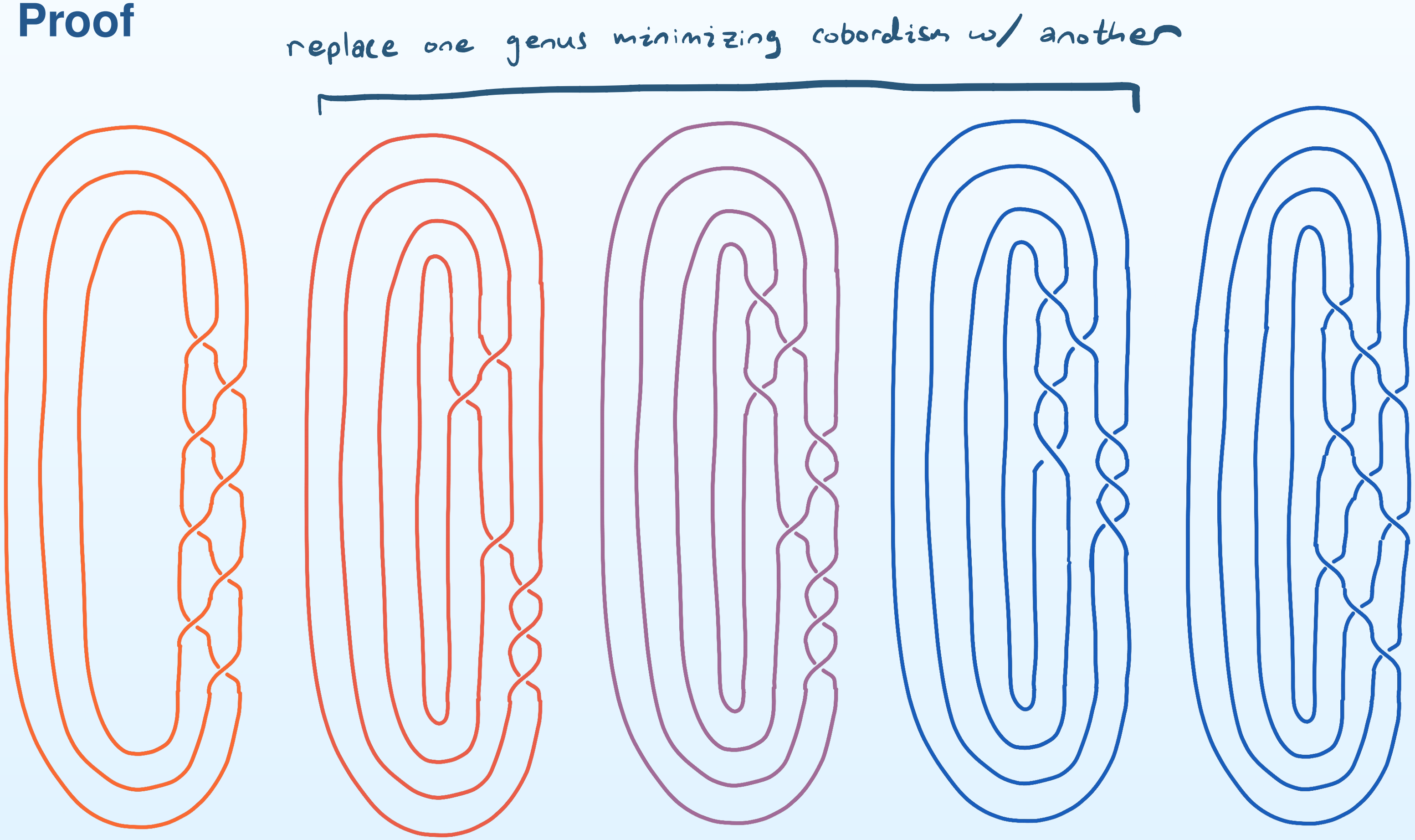

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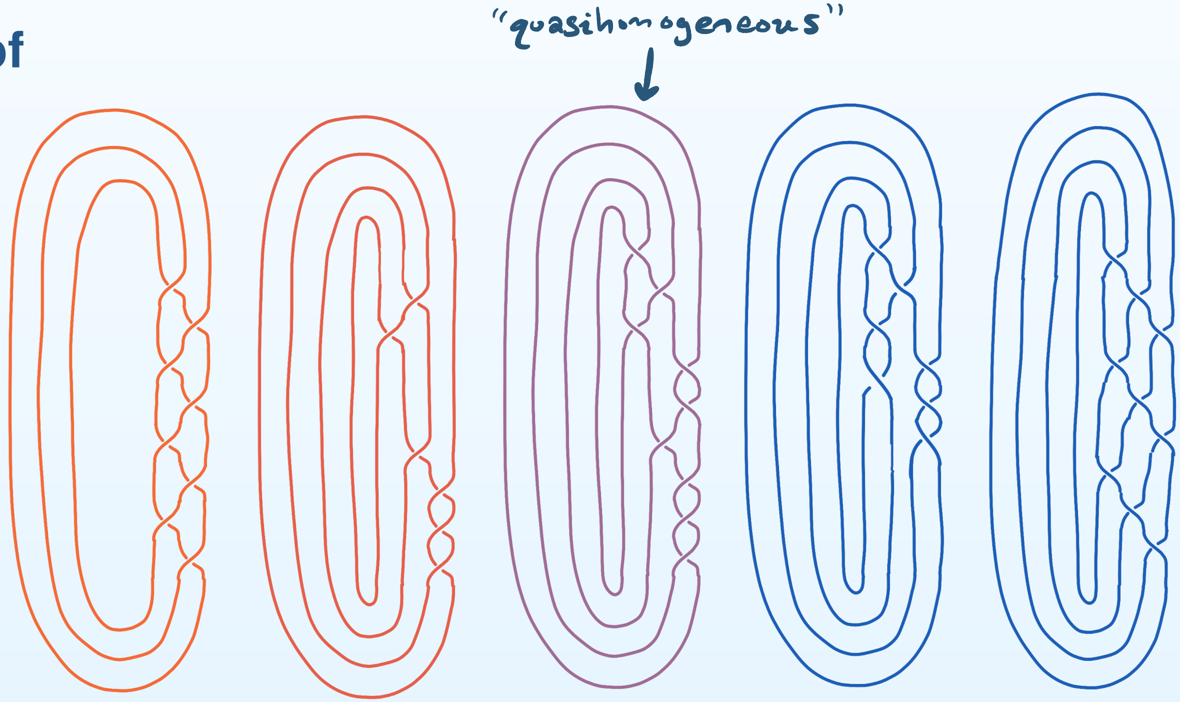

Using this lemma, Feller, Lewark, and Lobb prove that alternating and quasihomogenous knots are squeezed.

Ribbon Cobordisms

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In showing alternating knots are squeezed, we showed an arbitrary alternating knot resides in a ribbon cobordism

$$T_+ \xrightarrow{\mathscr{R}_1} A_+ \xrightarrow{\mathscr{R}_2} A \xrightarrow{\mathscr{R}_3} A_- \xrightarrow{\mathscr{R}_4} T_- \tag{1}$$

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Similarly, quasihomogenous knots live in a “co-span” of ribbon cobordisms

$$T_+ \xrightarrow{\mathscr{R}_+} Q \xleftarrow{\mathscr{R}_-} T_- \tag{2}$$

All squeezing cobordisms given by Feller, Lewark, and Lobb fall into one of these cases.

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$$(2) \qquad T_+ \xrightarrow{\mathscr{R}_+} Q \xleftarrow{\mathscr{R}_-} T_-$$

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Question

Are all squeezed knots contained in cobordisms of forms 1 or 2?

I. The Milnor Conjecture

II. Squeezed knots

III. Slice torus invariants and squeezed knots

- Slice torus invariants
- Stable invariants
- The Feller-Lewark-Lobb theorems
- $\ell(K)$ is well-defined
- K squeezed $\Rightarrow \ \ell(K) = -\ell(-K)$
- $\{\phi(K)\}_{\phi \in \mathcal{ST}} = [-\ell(-K), \ell(K)]$

IV. Obstructions

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III. Slice torus invariants and squeezed knots

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Slice torus invariants

Only known method for proving a knot is squeezed is to exhibit a genus minimizing cobordism.

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V. Future directions

Slice torus invariants

Only known method for proving a knot is squeezed is to exhibit a genus minimizing cobordism.

Let $\phi : \mathcal{C}_s \to \mathbb{R}$ be a group homomorphism.

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Denote the collection of all slice torus invariants by $\mathcal{ST}$.

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Livingston defines the **stable 4-ball genus** to be

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$$\widehat{g}_4(K) = \lim_{n\to\infty} \frac{g_4(\#^n K)}{n} = \lim_{n\to\infty} \frac{g_4(\overbrace{K\#K\#\cdots\#K}^{n\text{-fold connect sum}})}{n}$$

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Feller, Lewark, and Lobb define the **ℓ-invariant** of K to be

$$\ell(K) = \lim_{p \to \infty} \widehat{g}_4(K \# T_{p,p+1}) - \widehat{g}_4(T_{p,p+1})$$

The Feller-Lewark-Lobb theorems

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Squeezed knot valuation theorem (Feller, Lewark, Lobb '21)

If K is squeezed, then $\ell(K) = -\ell(-K)$.

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The set of all values in $\mathbb{R}$ taken on by K under slice torus invariants is $[-\ell(-K), \ell(K)]$.

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Corollary
All slice torus invariants agree on squeezed knots.

$\ell(K)$ is well-defined

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Lemma (Livingston '10)
$\widehat{g_4}$ is well-defined and induces a seminorm on the topological vector space $\mathcal{C}_s \otimes \mathbb{R}$.

This implies

$$t_p(K) := \widehat{g_4}(K \# T_{p,p+1}) - \widehat{g_4}(T_{p,p+1}) \geq 0.$$

So ℓ will be well-defined if we can show $t_p(K)$ is monotone decreasing.

$\ell(K) = \lim_{p\to\infty} \widehat{g}_4(T_{p,p+1}(K) - \widehat{g}_4(T_{p,p+1})$

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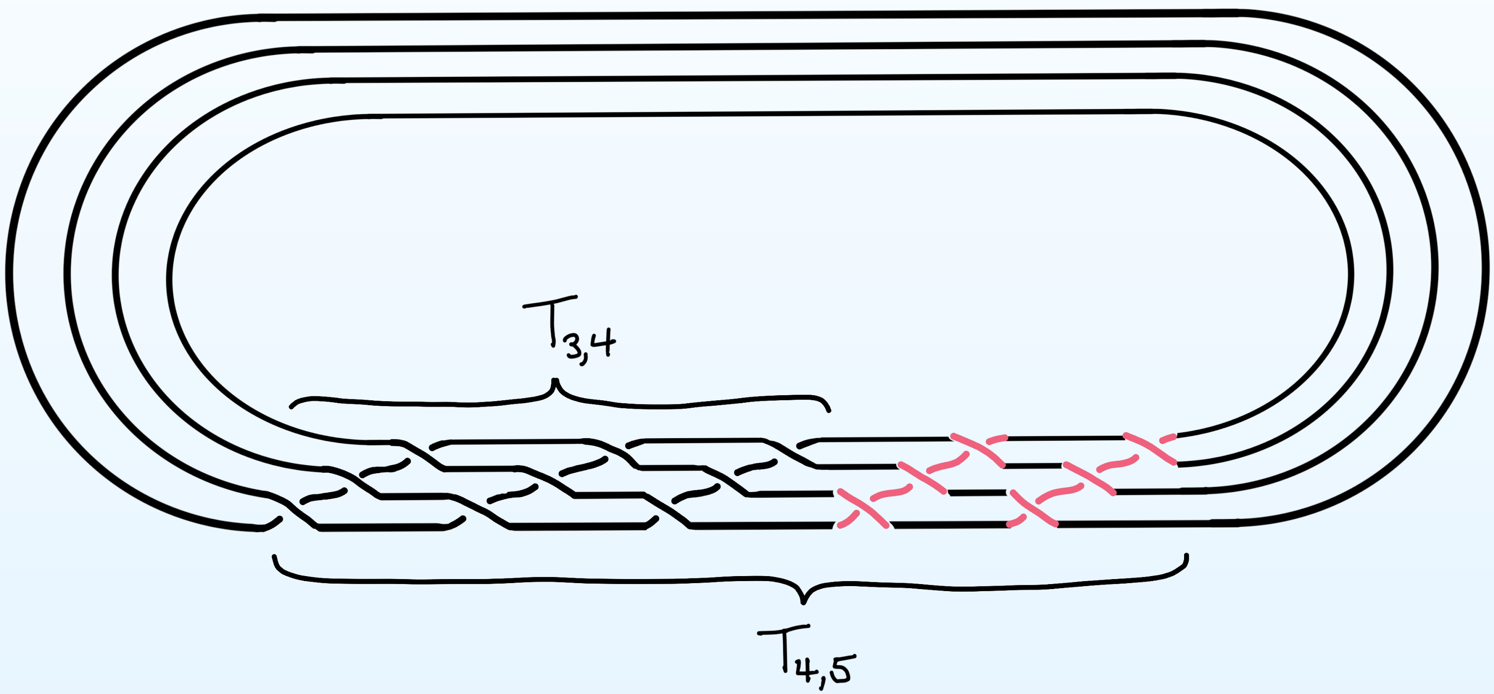
Recall $T_{p-1,p}$ is the closure of $\beta = (\sigma_1\sigma_2\cdots\sigma_{p-1})^{p-1}$



Adding $2(p-1)$ bands gives a cobordism

$$C : T_{p-1,p} \to T_{p,p+1} \qquad g(C) = \frac{2(p-1)}{2} = p - 1$$

$\ell(K) = \lim_{p\to\infty} \widehat{g}_4(T_{p,p+1}(K) - \widehat{g}_4(T_{p,p+1})$

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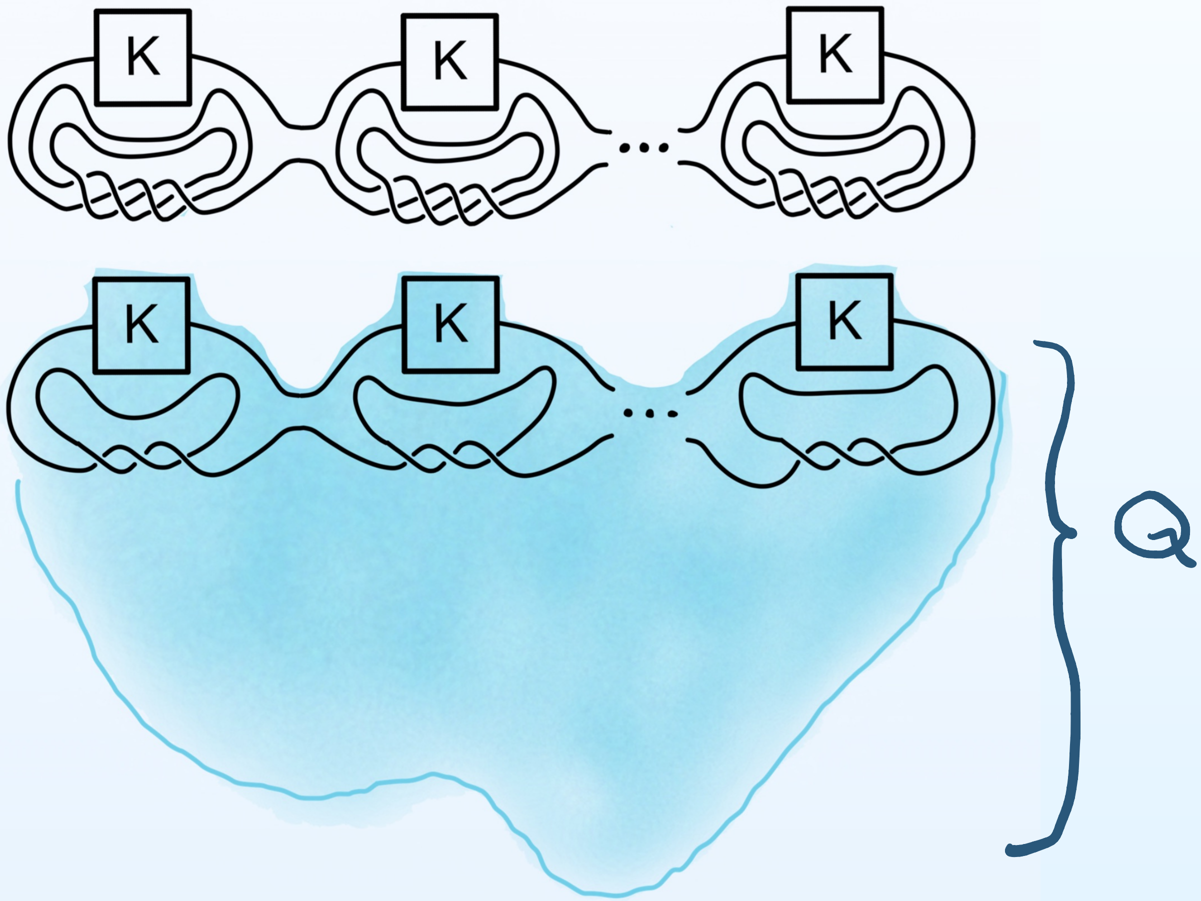
Let Q be a genus minimizing cobordism from $\#^n(T_{p-1,p}\#K)$ to the unknot.



$\ell(K) = \lim_{p\to\infty} \widehat{g}_4(T_{p,p+1}(K) - \widehat{g}_4(T_{p,p+1})$

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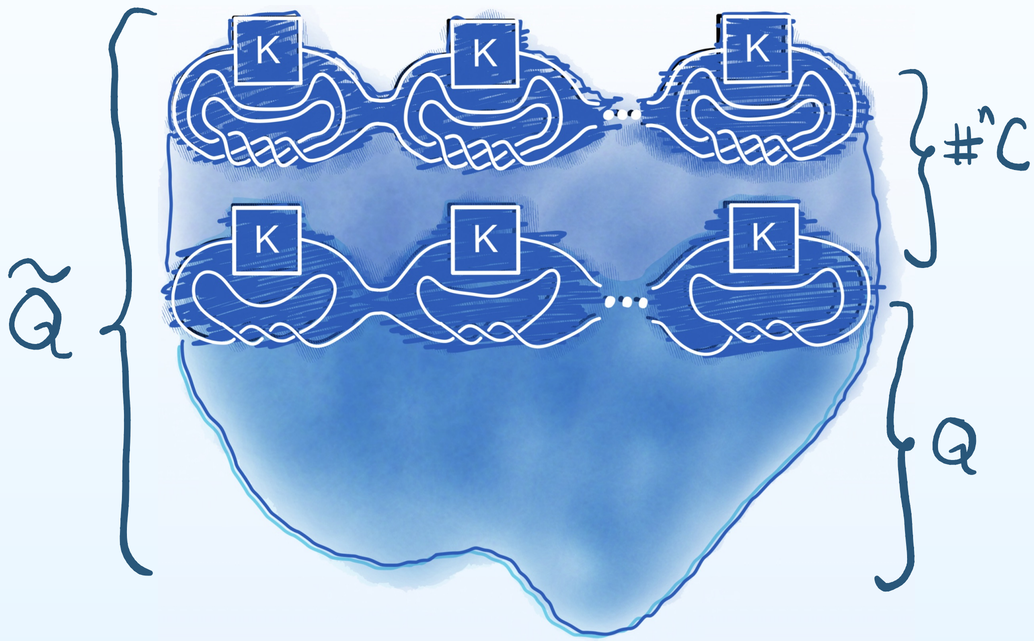

Gluing $\#^n C$ to Q gives a cobordism $\widetilde{Q} : \#^n(T_{p,p+1}\#K) \to U$

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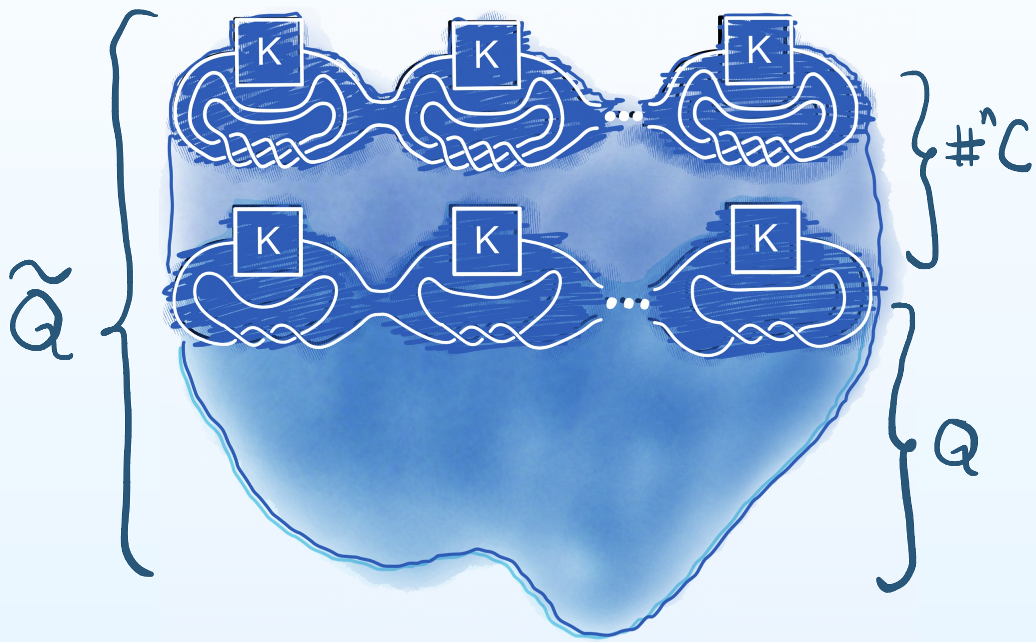

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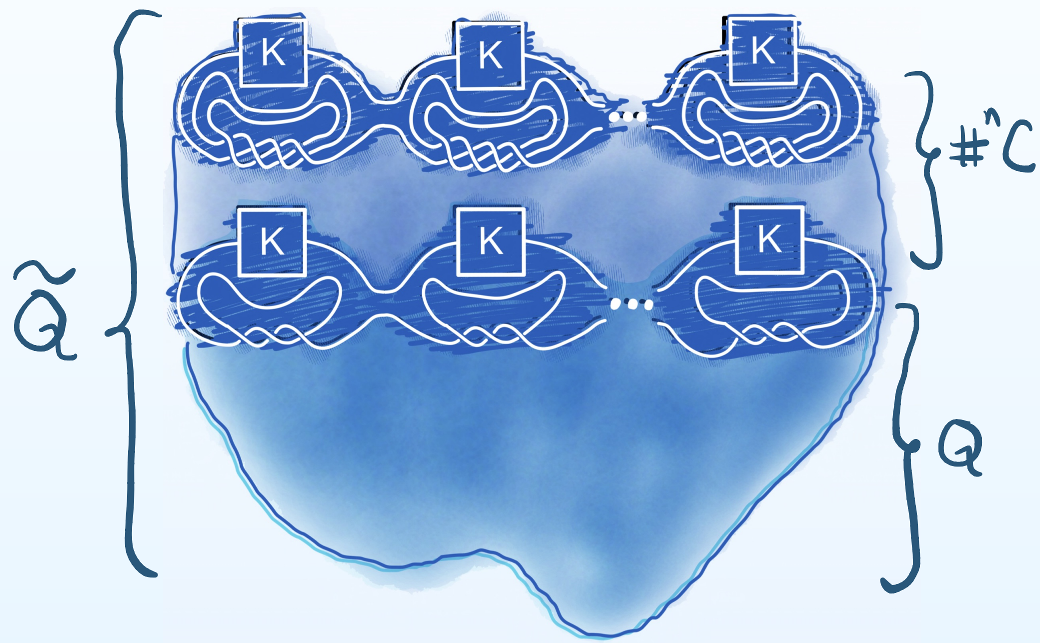

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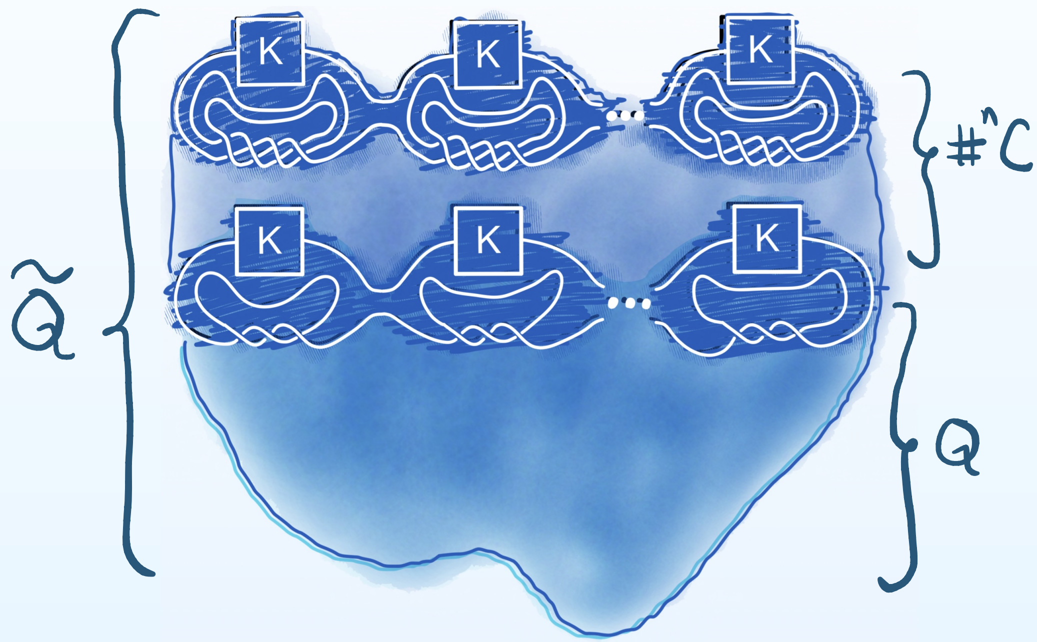

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Inequality is preserved under limits:

$$\begin{aligned} t_p(K) = & \lim_{n\to\infty} \frac{g_4(\#^n(T_{p,p+1}\#K))}{n} - \widehat{g}_4(T_{p,p+1}) \\ \leq & \lim_{n\to\infty} \frac{g_4(\#^n(T_{p-1,p}\#K)))}{n} - \widehat{g}_4(T_{p-1,p}) = t_{p-1}(K) \end{aligned}$$

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Lemma If K is squeezed, there is a $p \in \mathbb{N}$ such that K squeezed between $T_{p,p+1}$ and $-T_{p,p+1}$.

Let

$$C_+ : T_{p,p+1} \to K \qquad C_- : K \to -T_{p,p+1}$$

be the cobordisms implicit in the lemma, then

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$$\underbrace{0 \leq \ell(K) - (-\ell(-K))}_{\ell(K)\geq 0} \leq \underbrace{t_p(K) + t_p(-K)}_{\substack{t_p(K) \text{ monotone decreasing} \\ \text{and } t_p(K)\to\ell(K)}}$$

K squeezed $\Rightarrow \ \ell(K) = -\ell(-K)$

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$$\le \underbrace{g(C_+) + g(C_-)}_{\widehat{g}_4(K)\le g_4(K)} - d_{cob}(T_{p,p+1}, -T_{p,p+1}) = 0 \quad \square$$

$\{\phi(K)\}_{\phi\in\mathcal{ST}} \subseteq [-\ell(-K), \ell(K)]$

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For any $[K] \in \mathcal{C}_s$ and any slice torus invariant ϕ, we have $\phi[K] \leq \widehat{g}_4(K)$.

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A similar procedure gives the upper bound of $\ell(K)$. $\square$

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Using Levine-Tristram signatures, Litherland proved that the positive torus knots are linearly independent in $\mathcal{C}_s \otimes \mathbb{Q}$.

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Topologizing $\mathcal{T}$ carefully, one can realize $\mathcal{T} \cap (\mathcal{C}_s \otimes \mathbb{Q})$ as a dense subset of $\mathcal{T}$.

We then complete f and $\widehat{g_4}$ over $\mathbb{R}$ to get a $\mathbb{R}$-valued functional $\widetilde{f}$ satisfying $\widetilde{f} \leq \widehat{g_4}|_{\mathcal{T}}$.

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The goal is to extend $\widetilde{f}$ to a linear functional F on $\mathcal{T}_K := \mathcal{T} + \langle [K] \rangle$ satisfying $F([K]) = \alpha$.

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Recall $\widehat{g}_4$ is a semi-norm on $\mathcal{C}_s$ and therefore a subadditive function. we conclude F may be extended to all of $\mathcal{C}_s \otimes \mathbb{R}$ by Hahn-Banach.

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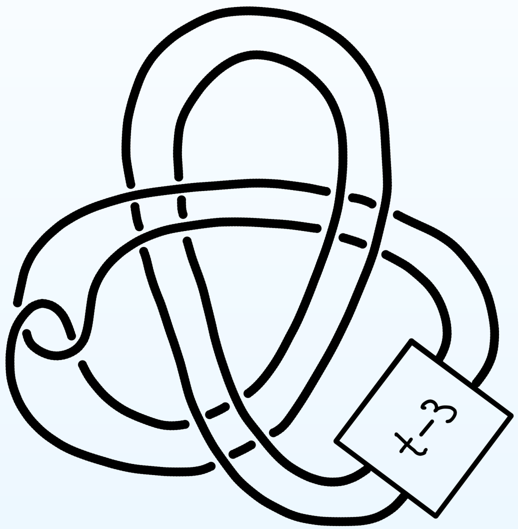
V. Future directions

Let $W^+(K,t)$ denote the t-twisted positive Whitehead double of K.



Obstructing Squeezed-ness with slice torus invariants

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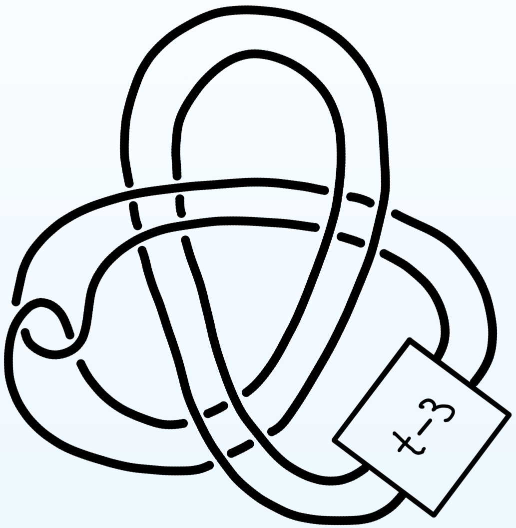

Theorem (Hedden and Ording '05)

$$\tau(W^+(T_{2,2n+1},t)) = \begin{cases} 0 & t > 2n-1 \\ 1 & t \leq 2n-1 \end{cases}$$

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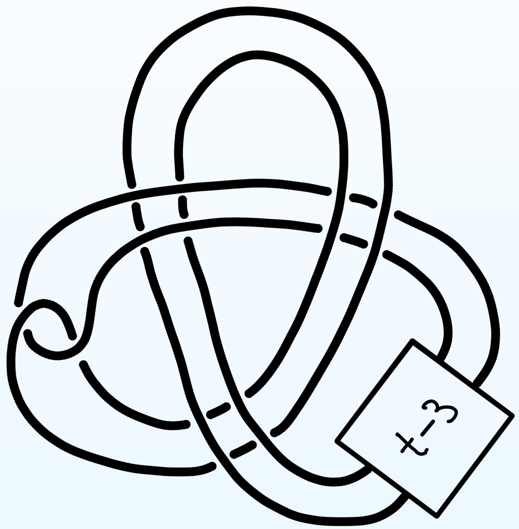

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Some computer computations revealed $\frac{s}{2} \neq \tau$ on the knots: $W^+(T_{2,5},5)$, $W^+(T_{2,5},4)$, $W^+(T_{2,7},7)$ and a few others.

Generalizing Hedden and Ording's results

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Theorem (Lewark-Zibrowius '22)
For P a pattern,[1] K a knot, and $\phi \in \mathcal{ST}$, there is a unique value $\vartheta_\phi(P, K) \in \mathbb{Z} \cup \infty$ such that

$$\phi(P(K,t)) = \phi(P(K,t-1)) - \begin{cases} 1 & t = \vartheta_\phi(P,K) \\ 0 & \text{else} \end{cases}$$

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For any $\phi \in \mathcal{ST}$, we have $TB(K) \leq \vartheta_\phi(W^+, K) \leq -TB(-K)$

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Question What is $\vartheta_s(W^+, T_{2,2n+1})$?

[1] with wrapping number 2 and winding number zero

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Note slice torus invariants are squeezing obstructions and that all squeezing obstructions are equal on squeezed knots!

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- Lipshitz and Sarkar found a suspension spectra for Khovanov homology.

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- Lipshitz and Sarkar found a suspension spectra for Khovanov homology.
- Consequently, Khovanov homology admits Steenrod squares

$$\mathrm{Sq}^n : \mathrm{Kh}(K) \xrightarrow{gr_h \mapsto gr_h + n} \mathrm{Kh}(K).$$

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$$r_-^\alpha(K) = -r_+^\alpha(-K) \qquad s_-^\alpha(K) = -s_+^\alpha(-K)$$

[2] or, more generally, any stable cohomology operation

9_{42}, 10_{132}, and 10_{136} are not squeezed

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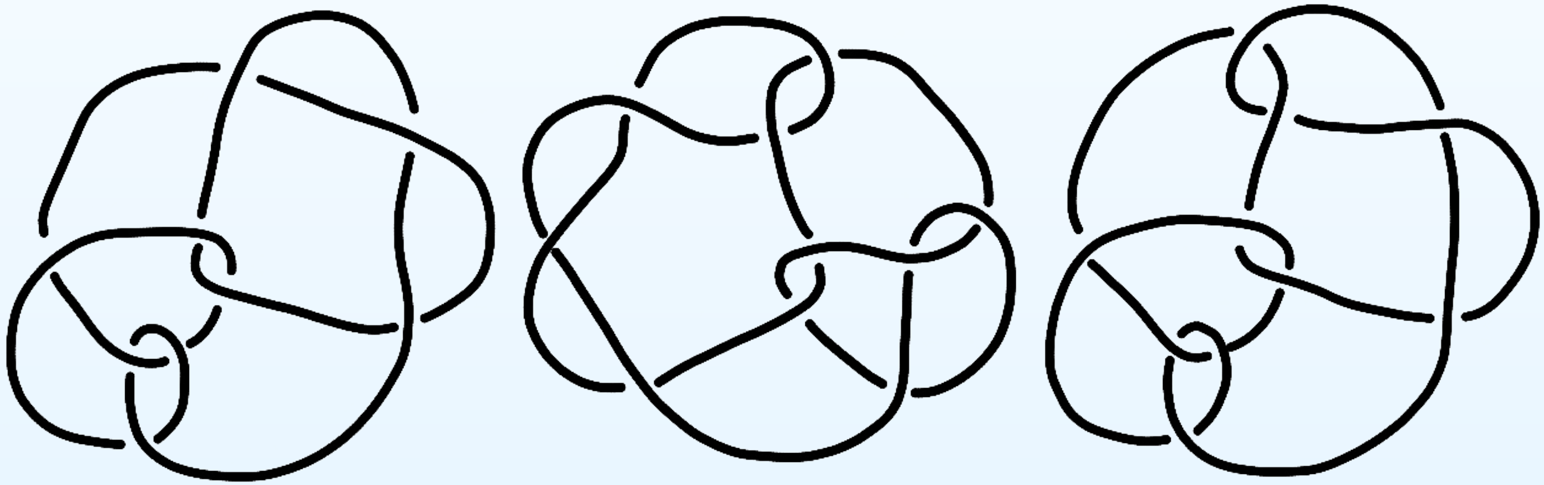
Lipshitz and Sarkar found several knots where $s_+^{\mathrm{Sq}^2}$ disagrees with $s^{\mathbb{F}_2}$. Their examples with $\leqslant 10$ crossings are:



9_{42}	10_{132}	10_{136}
$s^{\mathbb{F}_2} = 0$	$s^{\mathbb{F}_2} = -2$	$s^{\mathbb{F}_2} = 0$
$s_+^{\mathrm{Sq}^2} = 2$	$s_+^{\mathrm{Sq}^2} = 0$	$s_+^{\mathrm{Sq}^2} = 2$

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Questions I am curious about...

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- Are torsion generators of $\mathcal{C}_s$ (negative amphichiral knots in particular) all squeezed?

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  If K is a torsion generator, then $\widehat{g_4}(K) = 0 = \ell(K)$, so all slice torus invariants will agree on K.

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  How will it behave for general squeezed knots?

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THANK YOU!

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Relationship with the Turaev genus

To a diagram D, one may resolve all crossings as 0 or 1 to get naturally associated diagrams $D_{\vec{0}}$ and $D_{\vec{1}}$.

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To a diagram D, one may resolve all crossings as 0 or 1 to get naturally associated diagrams $D_{\vec{0}}$ and $D_{\vec{1}}$.

Capping off and performing band attachments, one forms the **Turaev surface of the diagram** D.

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$g_T(K)$ is fairly mysterious and has some interesting properties:

1. $g_T(K) = 0$ if and only if K is alternating
2. $g_T(K)$ provides a lower bound the Khovanov width and the knot Floer width

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Observation
In light of the Feller-Lewark-Lobb theorems, the strongest version of Jung-Kang-Kim's bound is

$$\ell(K) + \ell(-K) \leq g_T(K)$$

Questions
Why do slice torus invariants give bounds on the Turaev genus? Is this a coincidence, or is there a relationship between $\mathcal{C}_s$ and g_T?

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- The cobordism distance
- Squeezed Knots
- Species of squeezed knots
- Quasipositive knots
- Alternating knots
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- Ribbon Cobordisms
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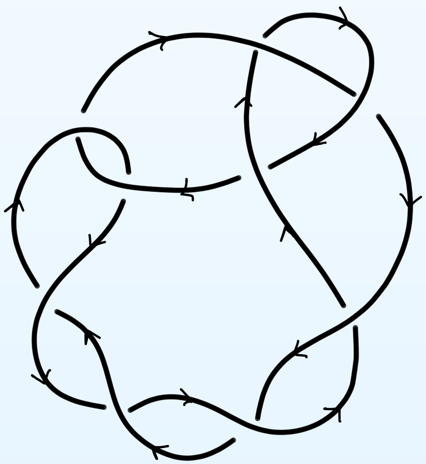
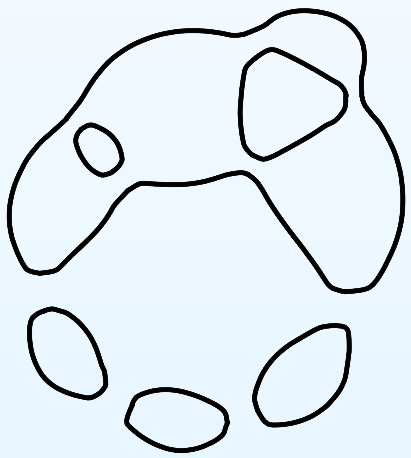
Proof D an alternating diagram D_o its oriented resolution





Alternating knots

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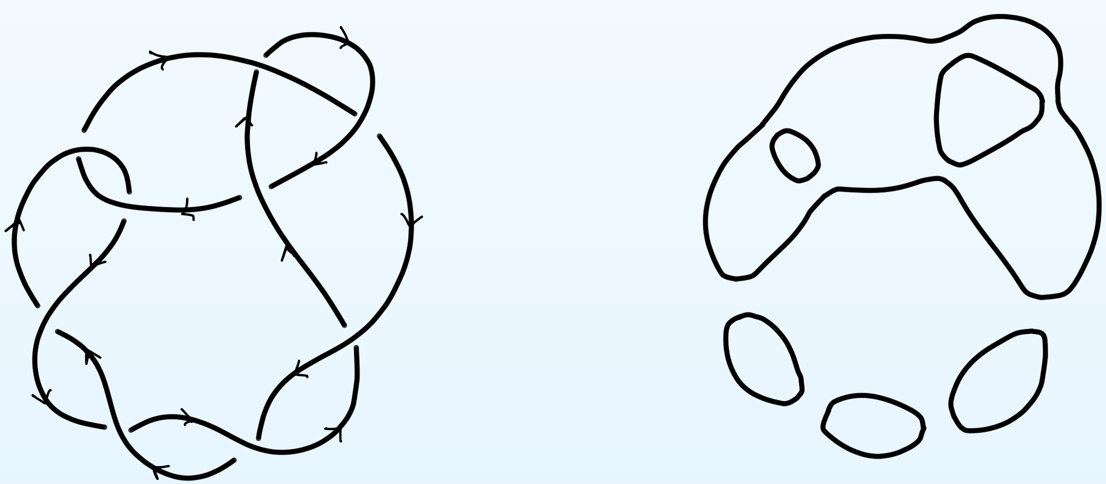

If $|\pi_0(D_o)| = N$, can choose $N-1$ crossings to connect all components of D_o, call these X_o.

I. The Milnor Conjecture
II. Squeezed knots
- The cobordism distance
- Squeezed Knots
- Species of squeezed knots
- Quasipositive knots
- Alternating knots
- Quasihomogenous knots
- Ribbon Cobordisms
III. Slice torus invariants and squeezed knots
IV. Obstructions
V. Future directions

Alternating knots

Build two diagrams

$$D_\pm = D_o \cup X_o \cup \underbrace{X_\pm}_{\text{all +/- crossings of } D}$$

The negative / positive crossings of $D_\pm$ will be nugatory, flip them to get a positive / negative link diagram.

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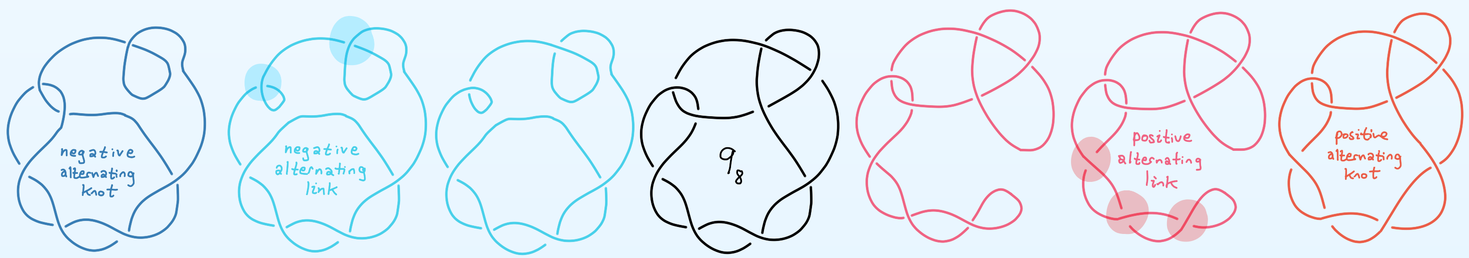

Add positive / negative bands to produce positive / negative alternating knots.

Alternating knots

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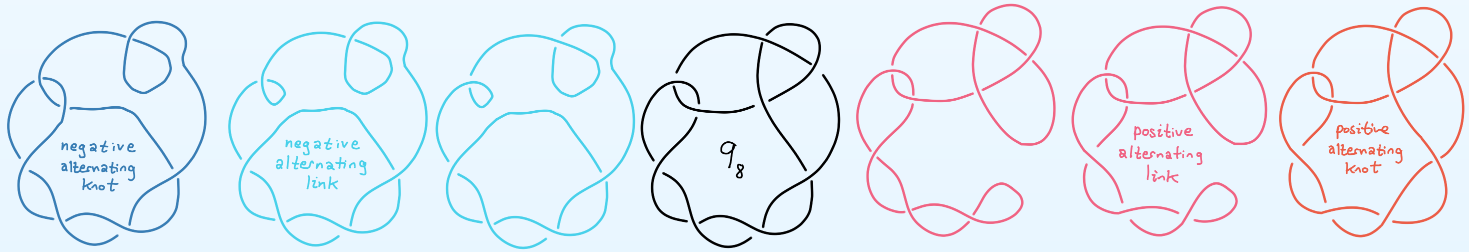

Add positive / negative bands to produce positive / negative alternating knots. A similar Euler characteristic computation to qpos. case confirms this cobordism minimizes genus.

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Quasihomogenous knots

Slogan: "A ribbon-immersed plumbing Σ of two ribbon-immersed surfaces $Q_{\pm}$ is similar to a Murasugi sum."

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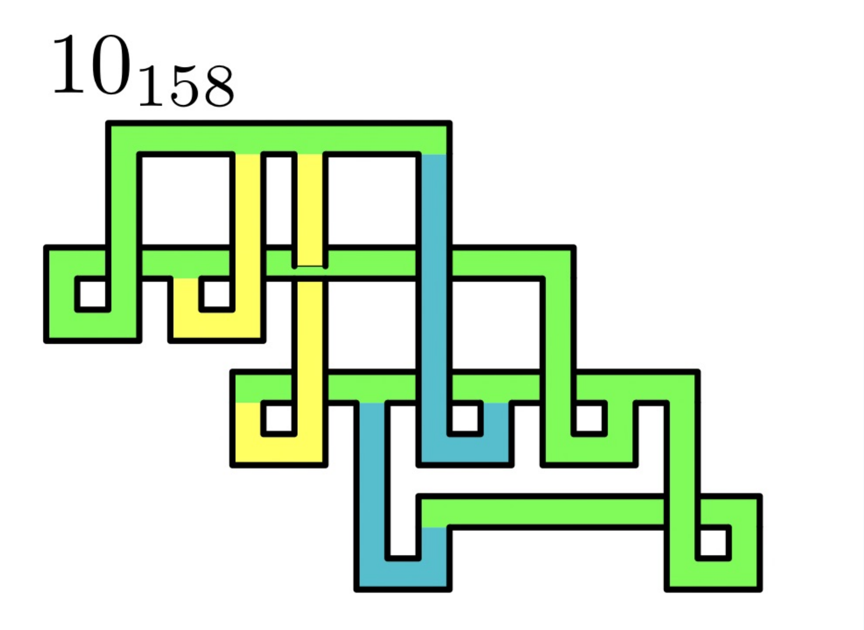

The crucial difference:

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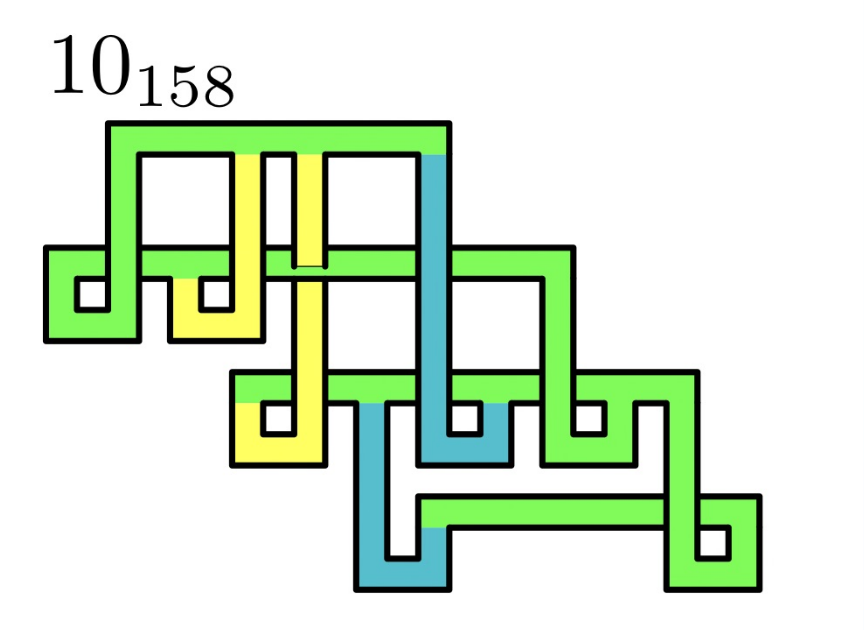

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We allow $Q_\pm$ to have ribbon singularities inside their respective copies of the identification region D,

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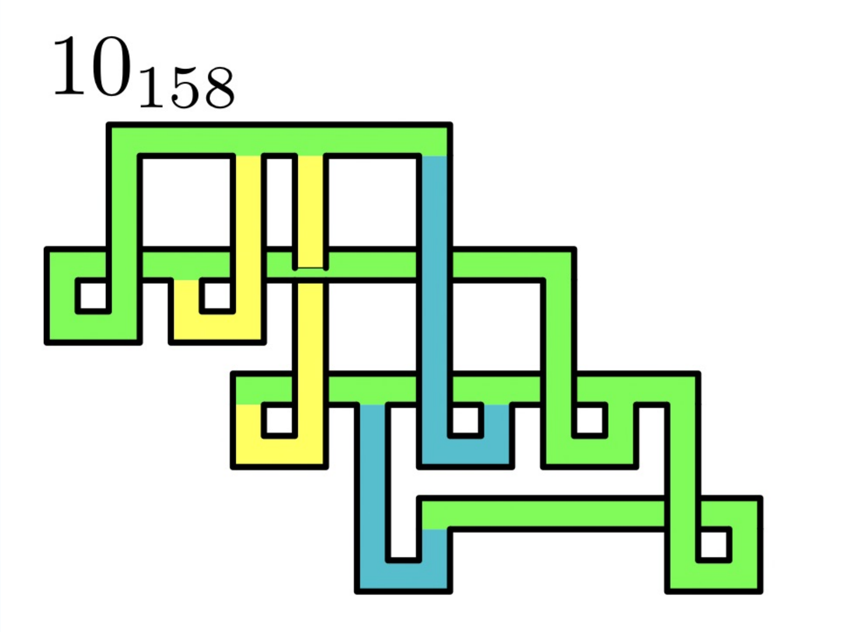

The crucial difference:
We allow $Q_\pm$ to have ribbon singularities inside their respective copies of the identification region D,but otherwise the images of $Q_\pm$ into Σ are disjoint.

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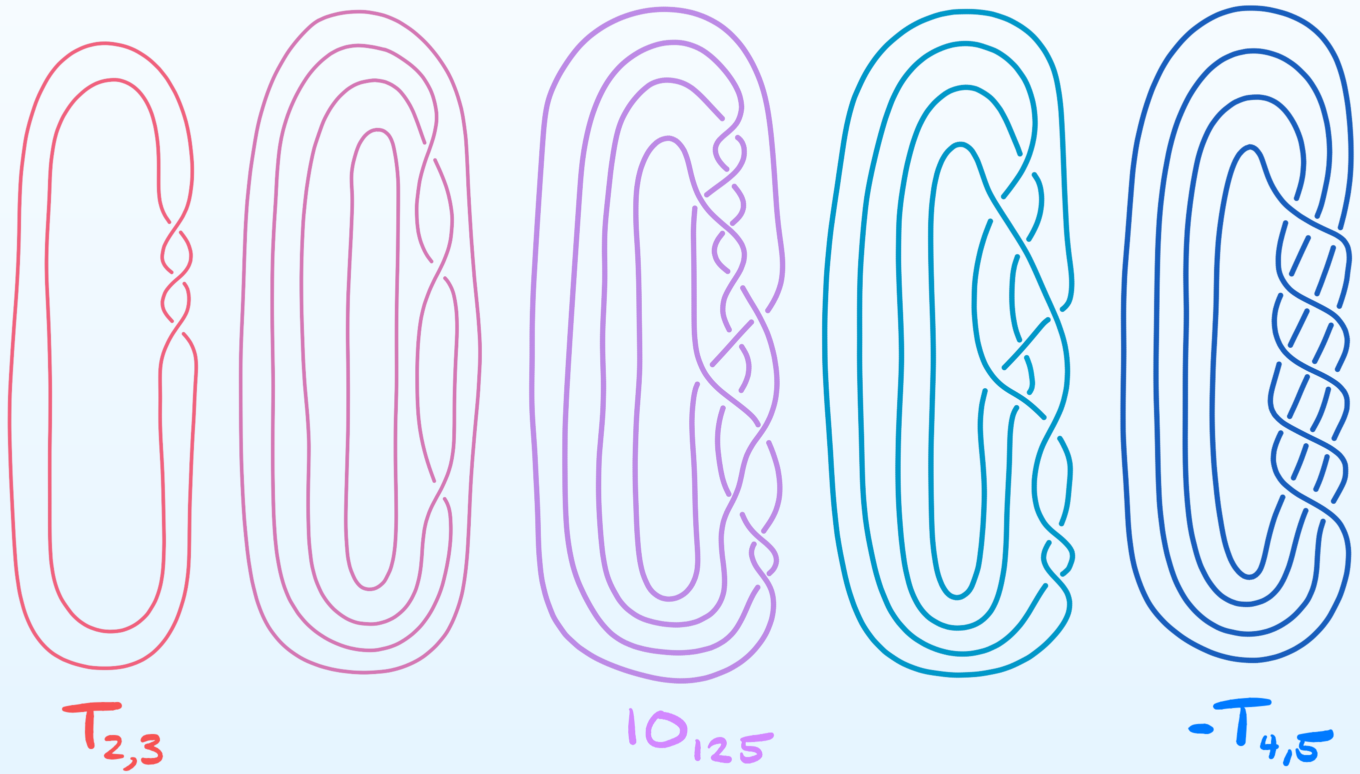
Quasihmogenous knots are squeezed

Proof follows from showing the cobordism from the immersed plumbing of the quasipositive/negative surfaces is genus minimzing.



ℓs is a squeezing obstruction

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● Obstructing Squeezed-ness with slice torus invariants
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Letting ℓs denote any of the Lipshitz-Sarkar refinements. Feller, Lewark, and Lobb prove that $\frac{\ell s}{2}$ is a squeezing obstruction.

Proof sketch

Lipshitz and Sarkar prove that $|\ell s(K) - s(K)| \in \{0, 2\}$ and that

$$|\ell s(K) - \ell s(J)| \leq 2d_{cob}(K, J)$$

so one needs to verify $\ell s(T_{p,q}) = 2g_4(T_{p,q})$.

Theorem (Lipshitz-Sarkar '05)

If for some $n \in \mathbb{N}$ we have $\mathrm{Sq}^n : \mathrm{Kh}(K) \xrightarrow{gr_h \mapsto gr_h + n} \mathrm{Kh}(K)$ is the zero map, then $s_{\pm}^{\mathrm{Sq}^n}(K) = r_{\pm}^{\mathrm{Sq}^n} = s^{\mathbb{F}_2}$

ℓs is a squeezing obstruction

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For K a positive knot and $q \in \{s(K) \pm 1\}$, it turns out to be sufficient to show the following maps are zero

$$\mathrm{Sq}^{\mathrm{n}} : \mathrm{Kh}^{-n,q}(K) \to \mathrm{Kh}^{0,q}(K)$$

$$\mathrm{Sq}^{\mathrm{n}} : \mathrm{Kh}^{0,q}(K) \to \mathrm{Kh}^{n,q}(K)$$

Leveraging classical computations of $\mathrm{Kh}(T_{2,2n+1})$, Feller, Lewark, and Lobb prove that for any knot K arising from a positive braid word, one has

$$\mathrm{Kh}^{t,q}(K) \cong 0 \qquad t \neq 0, q \in \{s(K) \pm 1\}$$

Hence, the two relevant maps are zero and $\ell s(K) = s^{\mathbb{F}_2}(K)$.

$\square$