SPIN & SPIN^c: a creature double-feature

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Some motivation

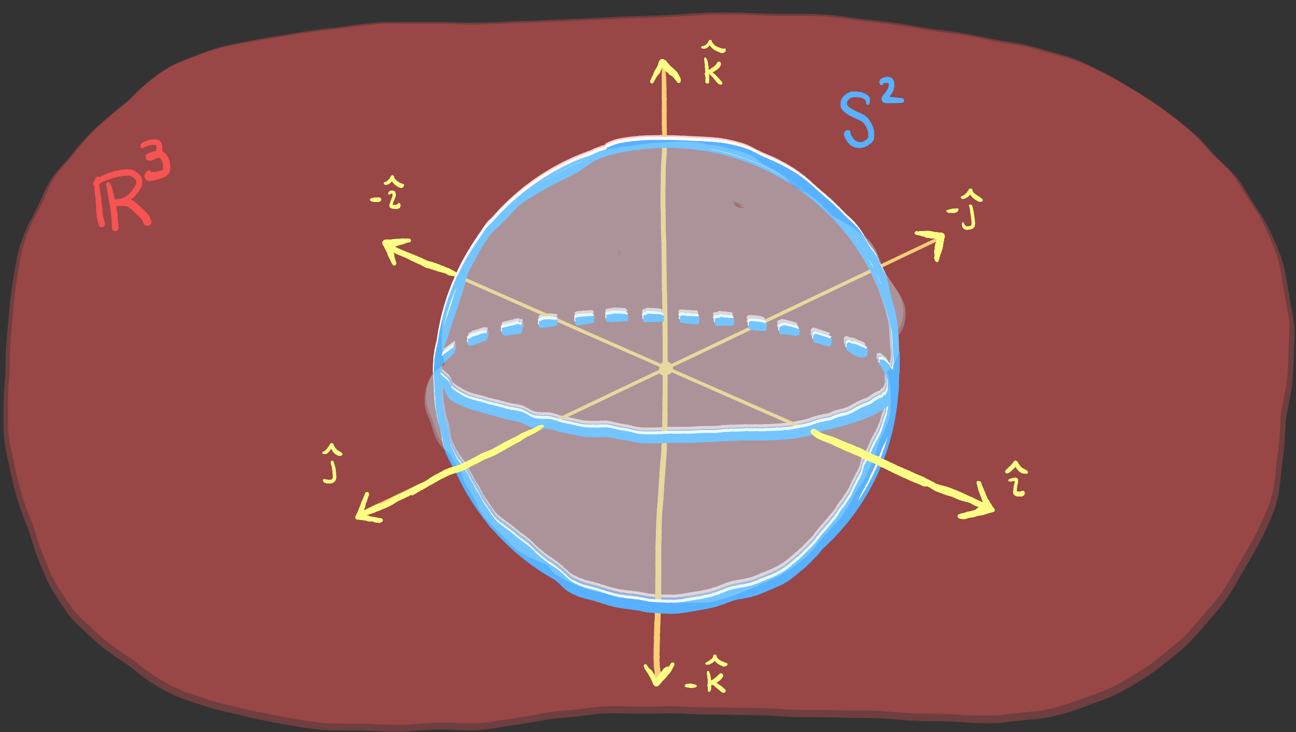
There are certain multiplicative groups which naturally produce rotation actions on $\mathbb{R}^n$, but also have "extra" information.

- $U(1) \cong \{e^{i\theta} \mid \theta \in [0, 2\pi)\} \overset{\text{"acts on"}}{\circlearrowright} \mathbb{R}^2 = \mathbb{C}$

  (action comes from complex multiplication)

- $SU(2) = \{\text{unit quaternions}\} \circlearrowright \mathbb{R}^3 = \{0 + a\hat{\imath} + b\hat{\jmath} + c\hat{k} \in \mathbb{H}\}$

  (action comes from conjugation)



What about higher dimensions?

I. SPIN(n)

Defn 1: Letting $n \geq 1$, Spin(n) is the smooth, double cover of SO(n).

"special orthogonal group"

When $n \neq 2$, we alternatively have:

Defn 2: Spin(n) is the unique Lie group such that

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(n) \longrightarrow SO(n) \longrightarrow 1$$

is exact.

BONUS: Spin(n) is a certain unital subgroup of the Clifford algebra for $\mathbb{R}^n$

Spin(n) as Covering Space

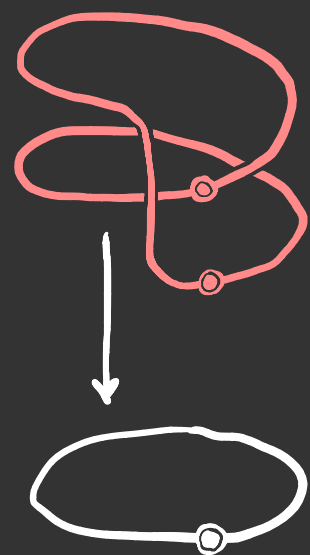
(n=1)

$SO(1)$ = trivial, has two-fold cover $\mathbb{Z}/2\mathbb{Z}$.

(n=2)

$SO(2) \cong S^1$, 2-fold cover is also S^1, so

$$\mathrm{Spin}(2) \cong S^1 \cong U(1)$$

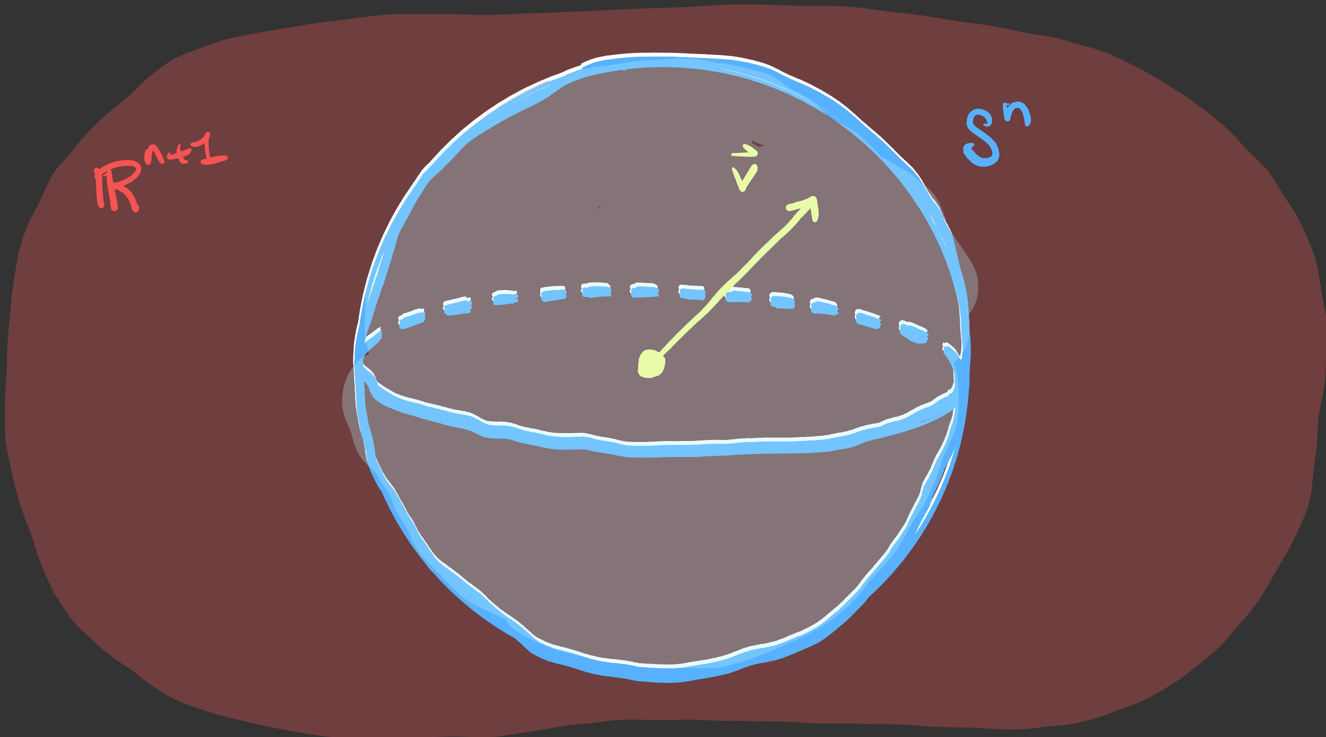


(n≥3) ... this takes a bit more work

__Lemma__

$$\forall n \geq 3: \quad \pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$$

Proof Embed S^n in $\mathbb{R}^{n+1}$ & choose $\vec{v} \in S^n$



(Gram-Schmidt)

Make an orthonormal basis whose final entry is $\vec{v}$. If $M\vec{v} = \vec{v}$, then

$$M = \begin{bmatrix} \begin{bmatrix} \text{something in } SO(n) \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \; 0 \; \cdots \; 0 & 1 \end{bmatrix}$$

fixes $\vec{v}$

Defn Let G be a Lie group. A principal G-bundle over a base space B is a fiber bundle

$$\pi : E \longrightarrow B$$

where $\pi^{-1}(x) \cong G$ for all $x \in E$.

Chasing definitions, we can realize $SO(n+1)$ as a principal $SO(n)$-bundle with base space S^n.

This yields a fibration:

$$SO(n) \xrightarrow{i} SO(n+1) \xrightarrow{\pi} S^n$$

(inclusion of fiber) (fiber bundle)

We can associate to the fibration a homotopy long exact sequence, which contains

$$
\begin{tikzcd}
\cdots \arrow[r] & \pi_2(S^n) \arrow[r] & \pi_1(SO(n)) \arrow[d] & \\
& & \pi_1(SO(n+1)) \arrow[r] & \pi_1(S^n) \arrow[r] & \cdots
\end{tikzcd}
$$

Fact 1: $\pi_i(S^j) = 0$ when $0 < i < j$

Fact 2: $SO(3) \cong \mathbb{RP}^3$

Fact 3: $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/2\mathbb{Z}$

So we have

$$\begin{tikzcd}
\cdots \arrow[r] & \underset{\pi_2(S^n)}{\overset{F1}{0}} \arrow[r] & \pi_1(SO(n)) \arrow[d, "\cong"] & & \\
& & \pi_1(SO(n+1)) \arrow[r] & \underset{\pi_1(S^n)}{\overset{F1}{0}} \arrow[r] & \cdots
\end{tikzcd}$$

Setting $n = 3$, we get

$$\begin{tikzcd}
\cdots \arrow[r] & \overset{F1}{0} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \ \ \text{F2 \& F3} \arrow[d, "\cong"] & & \\
& & \pi_1(SO(4)) \arrow[r] & \overset{F1}{0} \arrow[r] & \cdots
\end{tikzcd}$$

Repeating for each $n \geq 3$, we conclude

$$\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and}$$

SO(n) has a 2-sheeted cover!

Theorem Spin(n) is a Lie group

Proof sketch Denote the covering projection by

$$\pi : \mathrm{Spin}(n) \longrightarrow SO(n)$$

Spin(n) a smooth manifold follows from it being a smooth double cover of $SO(n)$.

For the group structure, $\forall g \in SO(n)$ we have $|\pi^{-1}(g)| = 2$

In particular,

$$\pi^{-1}(Id) = \{ Id^+, Id^- \}$$

which satisfy

$$Id^+ \cdot Id^+ = Id^+, \quad Id^+ \cdot Id^- = Id^-, \quad Id^- \cdot Id^- = Id^+$$

Associativity and closure under operation are inherited from $SO(n)$.

Spin Structures

- Let (M, g) be a Riemannian manifold (smooth, oriented). M has a canonical vector bundle called the tangent bundle TM.
- The metric g associates to TM the bundle of orthonormal (oriented) frames, $Fr(M)$, whose fiber over an $x \in M$ is the space of ordered, orthonormal bases $(\vec{v}_1, \ldots, \vec{v}_n)$.
- $Fr(M)$ is a principle $SO(n)$ bundle, so each fiber is identically a copy of $SO(n)$.

Def^n A spin structure on a manifold M is a principal spin(n)-bundle which is an "equivariant lift" of $Fr(M)$.

When do these exist?

Fact Every oriented, Riemannian n-manifold supports the principal SO(n)-bundle described in the last slide.

... but there is an obstruction to replacing each fiber with Spin(n)!

Theorem (Haefliger 1956)

An oriented Riemannian manifold M has a spin structure iff the second Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ vanishes.

II. $Spin^c(n)$

We get something incredibly useful when we extend spin(n) to its "complex analogue."
We can define $spin^c(n)$ as the 2-fold cover of $U(1) \times SO(n)$ or, using previous work, declare:

$$Spin^c(n) := U(1) \times spin(n) \big/ \mathbb{Z}/2\mathbb{Z}$$

Note $U(1)$ has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$, likewise spin(n) has the element we called $\overline{Id}$ which generates another $\mathbb{Z}/2\mathbb{Z}$.
This quotient operation glues the factors along their common $\mathbb{Z}/2\mathbb{Z}$.

We say an oriented, Riemannian n-manifold M has a spin^c-structure

if there is

- a principal $U(1)$-bundle, $E_{U(1)}(M)$
- a principal spin$^c(n)$-bundle, $E_{Spin^c}(M)$

such that we have a spin^c-equivariant map:

$$\xi : E_{Spin^c}(M) \longrightarrow E_{U(1)}(M) \times Fr(M)$$

FUN FACT i

- if a manifold M admits a spin structure, then it admits a spin^c structure

  $\downarrow$

  Take fiber product with a trivial $U(1)$-bundle.

Fun Fact ii

For some even dimensional manifolds, one can define an almost-complex structure J: a smooth $(1,1)$-tensor field such that, as a TM-automorphism we have $J^2 = -Id$.

Theorem Every almost-compex structure corresponds canonically with a particular spin^c structure.

Proof Sketch (Mellor)

Let $j: U(k) \longrightarrow SO(2k)$ be the "expansion" $i \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ homomorphism, this induces a map

$$g: U(k) \longrightarrow SO(2k) \times U(1)$$

$$g(M) := (j(M), \det(M))$$

g has a lift γ going to $Spin^c(2k)$ and M almost-complex means M has a unitary frame bundle $E_{U(n)}(TM)$, so we get a bundle

$$E_{Spin^c}(M) = E_{U(n)}(TM) \times_{\gamma} Spin^c(2k)$$

$\blacksquare$

These also turn out to be extremely useful structures in dimension 4.

Theorem (Vogt & Teichner)

All 4-manifolds admit a $spin^c$ structure.

Why you might care

- almost-complex manifolds are the settings for Floer homology theories which count equivalence classes of so-called pseudoholomorphic curves.

- this is the starting point for the powerful Seiberg-Witten invariants

References

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- Blake Mellor – "Spin^c- Manifolds"
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Thank

you.